Uniform labelled calculi for preferential conditional logics based on neighbourhood semantics

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Abstract

The preferential conditional logic $\mathbb{PCL}$, introduced by Burgess, and its extensions are studied. First, a natural semantics based on neighbourhood models, which generalises Lewis' sphere models for counterfactual logics, is proposed. Soundness and completeness of $\mathbb{PCL}$ and its extensions with respect to this class of models are proved directly. Labelled sequent calculi for all logics of the family are then introduced. The calculi are modular and have standard proof-theoretical properties, the most important of which is admissibility of cut, that entails a syntactic proof of completeness of the calculi. By adopting a general strategy, root-first proof search terminates, thereby providing a decision procedure for $\mathbb{PCL}$ and its extensions. Finally, semantic completeness of the calculi is established: from a finite branch in a failed proof attempt it is possible to extract a finite countermodel of the root sequent. The latter result gives a constructive proof of the finite model property of all the logics considered.

1 Introduction

Conditional logics have been studied from a philosophical viewpoint since the 1960s, with seminal works by, among other, Lewis, Nute, Stalnaker, Chellas, Pollock and Burgess.[1] In all cases, the aim is to represent a kind of hypothetical implication $A > B$ different from classical material implication, but also from other non-classical implications, such as the intuitionistic one.

There are mainly two kinds of interpretations of a conditional $A > B$. The first is hypothetical/counterfactual: "If A were the case then B would be the

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[1] Cf. [16], [27], [28], [3], [24], [2], [29].

case", while the second is prototypical: "Typically (normally) if A then B", or "B holds in most normal/typical cases in which A holds". Applications of conditional logics to computer science, more specifically to artificial intelligence and knowledge representation, have followed these two interpretations. The hypothetical/counterfactual interpretation has lead to the study of the relation of conditional logics with the notion of *belief change*, which involves the crucial issue of the Ramsey Test[2]. The prototypical interpretation has found an interest in the formalisation of default and non-monotonic reasoning (the well-known KLM systems) and has some relation with probabilistic reasoning. The range of conditional logics is actually more extensive, comprising also deontic and causal interpretations.

All interpretations of the conditional operator agree on the rejection of some properties of material implication (here denoted $\rightarrow$) along with properties of other non-classical implications, such as the intuitionistic one. Thus, the following properties:

Strengthening: $(A > B) \rightarrow ((A \wedge C) > B)$,

Transitivity: $((A > B) \wedge (B > C)) \rightarrow (A > C)$,

Contraposition: $(A > B) \rightarrow (\neg B > \neg A)$,

are rejected on the grounds that they impose constraints which are incompatible with a non-monotonic or counterfactual interpretation of the conditional.

The semantics of conditional logics is defined in terms of various kinds of possible-world models, most of them comprising a notion of preference, comparative similarity or choice among worlds. Intuitively, a conditional $A > B$ is true at a world x if B is true in all the worlds most normal/similar/close to x in which A is true. In contrast with the situation in standard modal logic, there is no unique semantics for conditional logics.

In this paper we consider the conditional logic $\mathbb{PCL}$ (Preferential Conditional Logic), one of the fundamental systems of conditional logics. An axiomatization of $\mathbb{PCL}$ (and the respective completeness proof) has been originally presented in the seminal work by Burgess in [2], where the system is called S, and then by Veltman [29].

The logic $\mathbb{PCL}$ generalises Lewis' basic logic of counterfactuals, and its flat fragment corresponds to the preferential logic P of non-monotonic reasoning proposed by Kraus, Lehmann and Magidor [15].

The logic takes its name, $\mathbb{PCL}$, from its original semantics, defined in terms of *preferential models*. In these models, every world x is associated with a set of accessible worlds W_x and a *preference* relation $\leq_x$ on this set; the intuition is that this relation assesses the relative normality/similarity of pairs of worlds with respect to x. Intuitively, a conditional $A > B$ is forced at x if B is true in

[2]The original formulation of the Ramsey test can be found in [25]. For its relationship with belief revision, refer to [6]. Moreover, the AGM theory of belief revision [1] has its origin in that work.

all accessible worlds (that is, worlds in W_x) where A holds and that are most "normal" with respect to x, where their normality is assessed by the relation $\leq_x$. According to some interpretations, normality is interpreted as minimality with respect to $\leq_x$.

In this paper we present an alternative semantics for $\mathbb{PCL}$ based on *neighbourhood models*. Neighbourhood semantics has been successfully employed to analyse non-normal modal logics [3], as their semantics cannot be defined in terms of usual Kripke models. In neighbourhood models, every world x is equipped with a set of neighbourhoods $N(x)$ and each $\alpha \in N(x)$ is a non-empty set of worlds. The general intuition is that each neighbourhood $\alpha \in N(x)$ represents a state of information/knowledge/affair to be taken into account in evaluating the truth of modal formulas at world x. In the conditional context, neighbourhood inclusion can be understood as follows: if $\alpha, \beta \in N(x)$ and $\beta \subseteq \alpha$, then worlds in β are at least as plausible/normal as worlds in α.

It turns out that neighbourhood models provide a very natural semantics for $\mathbb{PCL}$. This semantics abstracts away from the details of the preference relations and, moreover, the definition of the conditional can be seen as a simple modification of the *strict implication* operator, avoiding the unwanted properties of strengthening, transitivity and contraposition. The strict implication demands that each $\alpha \in N(x)$ "validates" the implication $A \to B$. The truth condition for the conditional only requires that, for all $\alpha \in N(x)$ containing an A -world, there is a smaller neighbourhood $\beta \subseteq \alpha$ non-vacuously validating the implication $A \to B$, where non-vacuously means that β must contain an A-world. No further properties or structure of neighbourhood models are needed.

The use of neighbourhood models for analysing conditional logics is not a novelty: Lewis' sphere models for counterfactual logics belong to this approach. However, the crucial property of sphere models is that neighbourhoods (e.g. spheres) are *nested*: given $\alpha, \beta \in N(x)$, either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$. This property entails that worlds belonging to $\bigcup N(x)$ can always be compared according to their level of normality[3]. This assumption is controversial in some contexts such as belief revision [9] and non-monotonic reasoning. The logic $\mathbb{PCL}$ is more general: its neighbourhood models do not assume nesting of neighbourhoods, whence worlds in $\bigcup N(x)$ are not necessarily comparable with respect to their level of normality.

Although $\mathbb{PCL}$ is the basic system we consider in this paper, stronger systems can be obtained by assuming properties of neighbourhood models: normality, total reflexivity, weak centering, centering, uniformity and absoluteness. These conditions are analogous to the ones considered by Lewis for sphere models, and give rise to a total of 15 preferential systems.

The Hilbert axiomatization of $\mathbb{PCL}$ is obtained by adding three simple axioms to the well-understood minimal conditional logics $\mathbb{CK}$ [3], as we will detail in Section 2. Notwithstanding the simplicity of the axiomatization and the fact

[3]In models where minimal spheres always exist, the nesting property is equivalent to the existence of a ranking function r_x defined for every world x. The function $r_x(y)$ evaluates the level of normality of each world $y \in W_x$ with respect to x.

that the axioms have been known for about forty year, the proof theory of $\mathbb{PCL}$ and its extensions remains largely unexplored. To the best of our knowledge, the only proof systems for $\mathbb{PCL}$ developed so far are those presented in [8, 26] and, more recently, in [18, 13]. All of them are based on preferential semantics, and the last two cover only the logic $\mathbb{PCL}$ and none of the extensions[4].

Building on neighbourhood semantics, we define labelled sequent calculi for $\mathbb{PCL}$ and its extensions. The calculi make use of both world and neighbourhood labels to encode the relevant features of the semantics into the syntax. All calculi are *standard*, meaning that each connective is handled exactly by dual left and right rules, justified through a clear meaning explanation. As a special feature, a new operator, $|$, is introduced for translating the meaning of the conditional operator into sequent rules. Moreover, the calculi are *modular*, to the extent that logical rules are the same for all systems, while relational rules for neighbourhood and world labels are added to define calculi for extensions. We do not consider explicitly the family of Lewis' logics, for which several internal and labelled calculi exist. Nonetheless the present framework can be adapted to cover these systems as well.

In addition to simplicity and modularity, the calculi have strong proof theoretical properties, such as height-preserving invertibility of all the rules and admissibility of contraction and cut.

We show that the calculi are terminating under the adoption of a uniform proof search strategy, obtaining thereby a decision procedure for (almost) all logics of the $\mathbb{PCL}$ family. However, since the logics in this family belong to different complexity classes [5], the uniform strategy will be unavoidably far from optimal.

We also prove semantic completeness of the calculus: from a failed proof search of a formula it is possible to extract a *finite* neighbourhood countermodel, built from a saturated branch of the attempted proof. This result provides a constructive proof of the finite model property for each logic of the $\mathbb{PCL}$ family with respect to the neighbourhood semantics.

The paper is organised as follows: In Section 2, the family of $\mathbb{PCL}$ logics and neighbourhood semantics is introduced. Section 3 shows completeness of $\mathbb{PCL}$ and its extensions with respect to neighbourhood semantics. In Section 4, we introduce labelled sequent calculi for family of preferential logics. In Section 5 we prove the main syntactic properties of the calculi, including admissibility of cut, thereby obtaining a syntactic proof of the their completeness. In Section 6, a decision procedure for the logics is presented. In Section 7, we present a proof of semantic completeness for the calculi, by extracting a countermodel form failed proof search. Finally, Section 8 discusses some related work.

[4]For a more detailed discussion on the literature, refer to Section 8.

2 Preferential logics and neighbourhood semantics

We introduce in this section the family of preferential conditional logics.

Definition 2.1. The set of well formed formulas of $\mathbb{PCL}$ and its extensions is defined as follows, for $p \in Atm$ a propositional variable and $A, B \in \mathcal{L}$:

$$\mathcal{L} ::= p \mid \bot \mid A \wedge B \mid A \vee B \mid A \to B \mid A > B.$$

Other propositional connectives and constants can be defined: $\neg A$ as $A \to \bot$, $A \leftrightarrow B$ as $(A \to B) \wedge (B \to A)$, and $\top$ as $p \vee \neg p$.

The axiomatic presentation of preferential conditional logics is obtained by adding to an axiomatization of classical propositional logic the axioms and inference rules in Figure 1. Inference rules (RCEA), (RCK) and axiom (R-And) added to an axiomatization of classical propositional logic constitute an axiomatization of the weakest conditional logic $\mathbb{CK}$ [3]. The addition of (ID), (CM) and (OR) to this system yields an axiomatization of the basic preferential conditional logic, $\mathbb{PCL}$. Logics extending $\mathbb{PCL}$ are obtained by adding to the basic system (combinations of) the axioms in Figure 1: (N), for *normality*, (T), for *total reflexivity*, (W), for *weak centering*, (C), for *centering*, (U_1) and (U_2), for *uniformity*, and (A_1) and (A_2), for *absoluteness*. These axioms give rise to 15 different logics, represented in the lattice of Figure 2.

Given a logic $\mathbb{PK}$, for $\mathbb{K}$ one of $\mathbb{CL}$, $\mathbb{N}$, $\mathbb{T}$, $\mathbb{W}$, $\mathbb{C}$, $\mathbb{U}$, $\mathbb{A}$, $\mathbb{NU}$, $\mathbb{TU}$, $\mathbb{WU}$, $\mathbb{CU}$, $\mathbb{NA}$, $\mathbb{TA}$, $\mathbb{WA}$, $\mathbb{CA}$, we denote by $\mathcal{H}_{\mathsf{PK}}$ the system of axioms of $\mathbb{PK}$, and by $\vdash_{\mathsf{PK}} F$ derivability of a formula F in $\mathcal{H}_{\mathsf{PK}}$. Thus, for instance, $\mathcal{H}_{\mathsf{PCL}}$ is the axiom system of $\mathbb{PCL}$, and $\vdash_{\mathsf{PCL}}$ is the derivability relation in $\mathcal{H}_{\mathsf{PCL}}$; similarly, $\mathcal{H}_{\mathsf{PN}}$ is the axiom system of $\mathbb{PN}$, and $\vdash_{\mathsf{PN}}$ is the derivability relation in $\mathcal{H}_{\mathsf{PN}}$.

The following proposition contains some theorems of $\mathbb{PCL}$ that will be (tacitly) used in the following. The first four are well-known axioms, respectively called (RT), (MOD), (DT), and (CSO) in the literature. Axiom (DT) is equivalent to (OR), and from (DT) axiom (RT) is derivable. Axiom (CSO) is equivalent to (CM)+(RT). The proof of the last three formulas can be found in [15].

Proposition 2.2. *The following formulas are derivable in* $\mathbb{PCL}$*:*

1. (RT) $\quad (A > B) \wedge ((A \wedge B) > C) \to (A > C)$*;*
2. (MOD) $\quad (A > \bot) \to (B > \neg A)$*;*
3. (DT) $\quad ((A \wedge B) > C) \to (A > (B \to C))$*;*
4. (CSO) $\quad (A > B) \wedge (B > A)) \to ((A > C) \to (B > C))$*;*
5. $((A \vee B) > A) \wedge ((B \vee C) > B) \to ((A \vee C) > A)$*;*
6. $((A \vee B) > A) \wedge ((B \vee C) > B) \to A > (C \to B))$*;*
7. $((A \vee B) > A) \wedge (B > C) \to (A > (B \to C))$.

(RCEA) $\dfrac{A \leftrightarrow B}{(A > C) \leftrightarrow (B > C)}$	(RCK) $\dfrac{A \to B}{(C > A) \to (C > B)}$
(ID) $A > A$	(R-And) $(A > B) \land (A > C) \to (A > (B \land C))$
(CM) $(A > B) \land (A > C) \to ((A \land B) > C)$	(OR) $(A > C) \land (B > C) \to ((A \lor B) > C)$
(N) $\neg(\top > \bot)$	(T) $A \to \neg(A > \bot)$
(W) $(A > B) \to (A \to B)$	(C) $(A \land B) \to (A > B)$
(U_1) $(\neg A > \bot) \to \neg(\neg A > \bot) > \bot$	(U_2) $\neg(A > \bot) \to ((A > \bot) > \bot)$
(A_1) $(A > B) \to (C > (A > B))$	(A_2) $\neg(A > B) \to (C > \neg(A > B))$

$$\mathcal{H}_{\mathsf{PCL}} = \{\text{(RCEA), (RCK), (ID), (R-And), (CM), (OR)}\};$$
$$\mathcal{H}_{\mathsf{PN}} = \mathcal{H}_{\mathsf{PCL}} + \text{(N)};\quad \mathcal{H}_{\mathsf{PT}} = \mathcal{H}_{\mathsf{PN}} + \text{(T)};\quad \mathcal{H}_{\mathsf{PW}} = \mathcal{H}_{\mathsf{PT}} + \text{(W)};\quad \mathcal{H}_{\mathsf{PC}} = \mathcal{H}_{\mathsf{PW}} + \text{(C)};$$
$$\mathcal{H}_{\mathsf{PU}} = \mathcal{H}_{\mathsf{PCL}} + (\mathrm{U}_1) + (\mathrm{U}_2);\quad \mathcal{H}_{\mathsf{PNU}} = \mathcal{H}_{\mathsf{PU}} + \text{(N)};\quad \mathcal{H}_{\mathsf{PTU}} = \mathcal{H}_{\mathsf{PNU}} + \text{(T)};$$
$$\mathcal{H}_{\mathsf{PWU}} = \mathcal{H}_{\mathsf{PTU}} + \text{(W)};\quad \mathcal{H}_{\mathsf{PCU}} = \mathcal{H}_{\mathsf{PWU}} + \text{(C)};$$
$$\mathcal{H}_{\mathsf{PA}} = \mathcal{H}_{\mathsf{PCL}} + (\mathrm{A}_1) + (\mathrm{A}_2);\quad \mathcal{H}_{\mathsf{PNA}} = \mathcal{H}_{\mathsf{PA}} + \text{(N)};\quad \mathcal{H}_{\mathsf{PTA}} = \mathcal{H}_{\mathsf{PNA}} + \text{(T)};$$
$$\mathcal{H}_{\mathsf{PWA}} = \mathcal{H}_{\mathsf{PTA}} + \text{(W)};\quad \mathcal{H}_{\mathsf{PCA}} = \mathcal{H}_{\mathsf{PWA}} + \text{(C)}.$$

Figure 1: Axiomatization of $\mathbb{PCL}$

$$\begin{tikzcd}
\mathbb{PA} \arrow[r] & \mathbb{PNA} \arrow[r] & \mathbb{PTA} \arrow[r] & \mathbb{PWA} \arrow[r] & \mathbb{PCA} \\
\mathbb{PU} \arrow[r] \arrow[u] & \mathbb{PNU} \arrow[r] \arrow[u] & \mathbb{PTU} \arrow[r] \arrow[u] & \mathbb{PWU} \arrow[r] \arrow[u] & \mathbb{PCU} \arrow[u] \\
\mathbb{PCL} \arrow[r] \arrow[u] & \mathbb{PN} \arrow[r] \arrow[u] & \mathbb{PT} \arrow[r] \arrow[u] & \mathbb{PW} \arrow[r] \arrow[u] & \mathbb{PC} \arrow[u]
\end{tikzcd}$$

Figure 2: The family of preferential conditional logics. An arrow between two systems $\mathbb{S}_1 \to \mathbb{S}_2$ means that $\mathbb{S}_2$ is an extension of $\mathbb{S}_1$. Extensions of $\mathbb{PCL}$ are denoted by $\mathbb{P}$ followed by the letter(s) corresponding to the axiom(s) added to the basic system: $\mathbb{N}$ for normality, $\mathbb{T}$ for total reflexivity, $\mathbb{W}$ for weak centering, $\mathbb{C}$ for centering, $\mathbb{U}$ for uniformity and $\mathbb{A}$ for absoluteness.

The semantics of $\mathbb{PCL}$ is usually defined in terms of preferential models, as explained in the Introduction. Here we define an alternative semantics in terms of neighbourhood models.

Definition 2.3. Let $\mathcal{C}l$ stand for *conditional logic* and $\mathcal{P}$ denote the powerset function. A *neighbourhood model* is a structure $\mathcal{M}_{\mathcal{C}l} = \langle W, N, [\![\ ]\!] \rangle$ where:

- W is a non empty set of elements, the *possible worlds*;
- $N : W \to \mathcal{P}(\mathcal{P}(W))$ is the *neighbourhood function*, which associates to

each $x \in W$ a family $N(x)$ of subsets of W, called a *system of neighbourhoods*;

- $[\![\ ]\!] : Atm \to \mathcal{P}(W)$ is the propositional evaluation.

The elements of $N(x)$ are called *neighbourhoods*, and are denoted by lowercase Greek letters. For all $x \in W$, we assume the neighbourhood function to satisfy the property of *non-emptiness*: For each $\alpha \in N(x)$, α is non-empty.

Notation 2.4. The symbol $\Vdash$ is used to denote the forcing (or truth) of a formula at a world of a model: $x \Vdash B$ means that B is true at x. Given a neighbourhood α, we use $\alpha \Vdash^{\exists} B$ as a shorthand for *there exists* $y \in \alpha$ *such that* $y \Vdash B$, and $\alpha \Vdash^{\forall} B$ as a shorthand for *for all* $y \in \alpha$*, it holds that* $y \Vdash B$.

Before giving its formal definition, we give an intuitive motivation of the truth condition for the conditional operator in neighbourhood semantics. Suppose we want to define a conditional operator more fine-grained than material implication, and suitable for an hypothetical, non-monotonic, or plausible interpretation. As a first attempt, we can define a kind of strict implication, in analogy to the corresponding notion in normal modal logic:

$$x \Vdash A > B \quad \textit{iff} \text{ for all } \alpha \in N(x) \text{ it holds } \alpha \Vdash^{\forall} A \to B. \tag{1}$$

However, this definition is not suitable for the conditional operator, as it would satisfy the unwanted properties of strengthening (or monotonicity), transitivity, and contraposition. An equivalent, slightly redundant, formulation of (1) consists in a restriction to neighbourhoods that contain A-worlds:

$$x \Vdash A > B \ \textit{iff} \text{ for all } \alpha \in N(x), \text{ if } \alpha \Vdash^{\exists} A \text{ then } \alpha \Vdash^{\forall} A \to B. \tag{2}$$

Thus, for every $\alpha \in N(x)$, if α contains an A-world, we require that $\alpha \Vdash^{\forall} A \to B$. The latter condition is too strong: in the intended interpretation, and in particular in the non-monotonic reading, the conditional should tolerate exceptions. Thus, instead of requiring $A \to B$ to be verified by the *whole* α, we only demand the formula to be verified by a sub-neighbourhood β of α.

$$\begin{aligned} x \Vdash A > B \quad \textit{iff} \quad & \text{for all } \alpha \in N(x), \text{ if } \alpha \Vdash^{\exists} A, \text{ then} \\ & \text{there exists a } \beta \in N(x) \text{ with } \beta \subseteq \alpha \text{ and} \\ & \text{such that } \beta \Vdash^{\forall} A \to B. \end{aligned} \tag{3}$$

Here, however, there is still a problem: the condition on β could be vacuously satisfied by choosing a β that does not contain any A-world (at least whenever $A \neq \top$). To rule out this case, we modify (3) as follows:

$$\begin{aligned} x \Vdash A > B \quad \textit{iff} \ & \text{for all } \alpha \in N(x), \text{ if } \alpha \Vdash^{\exists} A, \text{ then} \\ & \text{there exists } \beta \in N(x) \text{ with } \beta \subseteq \alpha \text{ and} \\ & \text{such that } \beta \Vdash^{\exists} A \text{ and } \beta \Vdash^{\forall} A \to B. \end{aligned} \tag{4}$$

Definition (4) is the truth definition of conditional adequate to formalize the logics of the preferential family.

Name	Acronym	Semantic condition	Axiom(s)
Normality	$\mathcal{N}$	For all $x \in W$ it holds that $N(x) \neq \emptyset$	(N)
Total reflexivity	$\mathcal{T}$	For all $x \in W$ there is $\alpha \in N(x)$ such that $x \in \alpha$	(T)
Weak centering	$\mathcal{W}$	For all $x \in W$ and $\alpha \in N(x)$, $x \in \alpha$	(W)
Centering	$\mathcal{C}$	For all $x \in W$ and $\alpha \in N(x)$, $x \in \alpha$ and $\{x\} \in N(x)$	(C)
Uniformity	$\mathcal{U}$	For all $x \in W$, if $y \in \alpha$ and $\alpha \in N(x)$, then $\bigcup N(x) = \bigcup N(y)$.	(U_1) + (U_2)
Absoluteness	$\mathcal{A}$	For all $x \in W$, if $y \in \alpha$ and $\alpha \in N(x)$, then $N(x) = N(y)$.	(A_1) + (A_2)

Figure 3: Semantic conditions to be added to $\mathcal{M}_{\mathcal{C}l}$, with corresponding axioms to be added to $\mathcal{H}_{\mathsf{PCL}}$.

Definition 2.5. The notion of *truth of a formula at a world of a model* is defined as follows, for $F \in \mathcal{L}$ formula, $\mathcal{M}_{\mathcal{C}l} = \langle W, N, [\![\ ]\!] \rangle$ neighbourhood model and $x \in W$ world:

- $x \Vdash p$ if $x \in [\![p]\!]$;
- $x \Vdash \neg A$ if $x \nVdash A$;
- $x \Vdash A \wedge B$ (resp. $A \vee B$) if $x \Vdash A$ and (resp. or) $x \Vdash B$;
- $x \Vdash A \to B$ if $x \Vdash \neg A$ or $x \Vdash B$;
- the truth condition for the conditional operator is (4).

We say that *a formula F is valid in $\mathcal{M}_{\mathcal{C}l}$* if for all $x \in W$, $x \Vdash F$. We say that *a formula F if valid in the class of all neighbourhood models* if for all neighbourhood models $\mathcal{M}_{\mathcal{C}l}$ it holds that F is valid in $\mathcal{M}_{\mathcal{C}l}$; this will be denoted by $\vDash_{\mathcal{C}l} F$.

Classes of neighbourhood models extending $\mathcal{M}_{\mathcal{C}l}$ are defined by adding to the definition of $\mathcal{M}_{\mathcal{C}l}$ one (or more) of the semantic conditions listed in Figure 3. More precisely, we define 14 classes of models extending $\mathcal{M}_{\mathcal{C}l}$: for $\mathcal{K}$ one of $\mathcal{N}$, $\mathcal{T}$, $\mathcal{W}$, $\mathcal{C}$, $\mathcal{U}$, $\mathcal{A}$, $\mathcal{NU}$, $\mathcal{TU}$, $\mathcal{WU}$, $\mathcal{CU}$, $\mathcal{NA}$, $\mathcal{TA}$, $\mathcal{WA}$, $\mathcal{CA}$, let $\mathcal{M}_{\mathcal{K}}$ be the class of neighbourhood models extending $\mathcal{M}_{\mathcal{C}l}$ with the corresponding semantic condition(s)[5].

The notion of truth of a formula at a world of a model $\mathcal{M}_{\mathcal{K}}$ is the same as in Definition 2.5. We say that *formula F is valid in $\mathcal{M}_{\mathcal{K}}$* if for all $x \in W$, $x \Vdash F$.

[5]The property we call *uniformity* is sometimes called *local uniformity*, to distinguish it from the following property of *strong uniformity*: for all $x, y \in W$, $\bigcup N(x) = \bigcup N(y)$. However, the set of valid formulas in the class of models satisfying (strong) uniformity and local uniformity is the same. A similar remark applies to the property of absoluteness.

We say that *formula F is valid in the class of neighbourhood models with $\mathcal{K}$* if for all neighbourhood models in the class it holds that F is valid in $\mathcal{M}_\mathcal{K}$; this will be denoted by $\models_\mathcal{K} F$.

Intuitively, given a world x of a neighbourhood model, we can think of $\bigcup N(x)$ as the set of all worlds accessible from x or, in other words, considered plausible from the viewpoint of x. Then, the semantic condition of *normality* says that x has some accessible worlds; *total reflexivity* that x is accessible to itself (i.e. x considers itself as plausible), and *uniformity* that, given some world y accessible form x, the set of worlds plausible for y is the same as the set of worlds plausible for x.

Absoluteness is a condition stronger than uniformity, and states that if y is accessible from x, then the neighbourhood systems of x and y are the same.

To understand the remaining conditions, think of a neighbourhood $\alpha \in N(x)$ as an epistemic state (or a proposition) described by means of a set of worlds, which are all equally plausible. Then, a neighbourhood $\beta \in N(x)$ such that $\beta \subseteq \alpha$ contains worlds that, from the viewpoint of x, are considered as "more plausible" than those in α, as they are contained in both neighbourhoods[6]. The condition of *weak centering* says that the "current" world x is part of any neighbourhood / state α, so that x is plausible in all states. *Centering* says that from the viewpoint of the current world x, the state containing just x is the "most plausible" state with respect to x. The conditions of total reflexivity, weak centering and centering are significant for the counterfactual interpretation of the conditional operator, as well as for its relations with belief change. On the contrary, they are not pertinent in the interpretation of the conditional for reasoning about typicality and normality, as in non-monotonic inferences.

Not all the extensions of $\mathbb{PCL}$ are proper conditional logics. We observe that

1. $\mathbb{PCA}$ collapses to classical propositional logic;
2. $\mathbb{PWA}$ collapses to $\mathsf{S5}$.

We provide a proof of the above through the semantics, obtaining a collapse of models. This implies the collapse of logical systems, once completeness has been proved.

For 1, we prove that $N(x) = \{\{x\}\}$. Let $y \in \alpha$ and $\alpha \in N(x)$. By absoluteness, $N(x) = N(y)$. By centering, $\{x\} \in N(x)$ and $\{y\} \in N(y)$, so that $\{y\} \in N(x)$ and $x \in \{y\}$, whence $x = y$. It follows that there is only one possible world, and the forcing condition of the conditional collapses to the one of material implication.

For 2, we prove that $N(x) = \{S\}$, where S is an arbitrary set of worlds containing x. Let $\alpha, \beta \in N(x)$. We show that $\alpha = \beta$. Let $y \in \alpha$; then, by absoluteness $N(x) = N(y)$, so $\beta \in N(y)$, and by centering $y \in \beta$. We conclude $\alpha \subseteq \beta$. The other inclusion is proved in the same way. Moreover, from the fact that for any $y \in S$, $N(y) = \{S\}$ it follows that all the possible worlds are equivalent: thus, the forcing condition of a conditional $A > B$ reduces to the truth condition of the strict implication $\Box(A \to B)$.

[6] These interpretation comes from Lewis' Sphere semantics [16]

By adding to $\mathcal{H}_{\mathsf{PCL}}$ the axiom

$$\text{(CV)} \quad ((A > C) \wedge \neg(A > \neg B)) \rightarrow ((A \wedge B) > C)$$

we obtain the logic $\mathbb{V}$, which is the basic system of Lewis' counterfactual logic. By adding the axiom to the other preferential logics, we get the family of counterfactual logics, $\mathbb{V}$ and extensions, introduced in [16]. Lewis defined the semantics of counterfactual logics in terms of sphere models; and sphere models for $\mathbb{V}$ can be obtained by adding to neighbourhood models the following condition:

Nesting: For all $\alpha, \beta \in N(x)$, either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$.

Thus, the family of Lewis' logics is by all means an extension of the preferential systems, and the proof theoretic and model theoretic methods detailed in the following sections can be (almost modularly) extended to cover Lewis' logics.

3 Soundness and completeness of neighbourhood models

We now prove soundness and completeness of the classes of models with respect to the axioms of $\mathbb{PCL}$ and its extensions.

3.1 Soundness

Theorem 3.1 (Soundness)**.** *For $F \in \mathcal{L}$, and $\mathbb{K}$ one of $\mathbb{CL}$, $\mathbb{N}$, $\mathbb{T}$, $\mathbb{W}$, $\mathbb{C}$, $\mathbb{U}$, $\mathbb{A}$, $\mathbb{NU}$, $\mathbb{TU}$, $\mathbb{WU}$, $\mathbb{CU}$, $\mathbb{NA}$, $\mathbb{TA}$, $\mathbb{WA}$, $\mathbb{CA}$, it holds that if $\vdash_{\mathsf{PK}} F$, then $\models_{\mathcal{K}} F$.*

Proof. The proof consists in showing that the axioms are valid, and that the inference rules preserve validity. By means of example, we prove soundness of axioms (CM), (OR) and (U_1).
(CM) $((A > B) \wedge (A > C)) \rightarrow ((A \wedge B) > C)$. Consider an arbitrary neighbourhood model $\mathcal{M}_{\mathcal{C}l}$ and an arbitrary world x, and suppose that x forces the antecedent of the implication. We show that x forces the succedent. The assumption means that:

1. $\mathcal{M}_{\mathcal{C}l}, x \Vdash A > B$, meaning that for each $\alpha \in N(x)$, if $\alpha \Vdash^{\exists} A$ then there exists $\beta \subseteq \alpha$ such that $\beta \Vdash^{\exists} A$ and $\beta \Vdash^{\forall} A \rightarrow B$;
2. $\mathcal{M}_{\mathcal{C}l}, x \Vdash A > C$, meaning that for each $\alpha \in N(x)$, if $\alpha \Vdash^{\exists} A$ then there exists $\gamma \subseteq \alpha$ such that $\gamma \Vdash^{\exists} A$ and $\gamma \Vdash^{\forall} A \rightarrow C$.

Suppose that there is $\alpha \in N(x)$ such that $\alpha \Vdash^{\exists} A \wedge B$; in particular, $\alpha \Vdash^{\exists} A$ so by 1 we have that there is $\beta \subseteq \alpha$ such that $\beta \Vdash^{\exists} A$ and $\beta \Vdash^{\forall} A \rightarrow B$. By 2 from $\beta \Vdash^{\exists} A$, we have that there is $\gamma \subseteq \beta$ such that $\gamma \Vdash^{\exists} A$ and $\gamma \Vdash^{\forall} A \rightarrow C$. Since $\gamma \subseteq \beta$ and $\gamma \Vdash^{\exists} A$, by $\beta \Vdash^{\forall} A \rightarrow B$ we get $\gamma \Vdash^{\exists} A \wedge B$. From $\gamma \Vdash^{\forall} A \rightarrow C$, a fortiori we have $\gamma \Vdash^{\forall} A \wedge B \rightarrow C$, so we have proved that $x \Vdash A \wedge B > C$.
(OR) $((A > C) \wedge (B > C)) \rightarrow ((A \vee B) > C)$. Suppose there is a neighbourhood model which satisfies the antecedent, i.e.

1. $\mathcal{M}_{\mathcal{Cl}}, x \Vdash A > C$, meaning that for each $\alpha \in N(x)$, if $\alpha \Vdash^{\exists} A$ then there exists $\alpha' \subseteq \alpha$ such that $\alpha' \Vdash^{\exists} A$ and $\alpha' \Vdash^{\forall} A \to C$;
2. $\mathcal{M}_{\mathcal{Cl}}, x \Vdash B > C$, meaning that for each $\beta \in N(x)$, if $\beta \Vdash^{\exists} B$ then there exists $\beta' \subseteq \beta$ such that $\beta' \Vdash^{\exists} B$ and $\beta' \Vdash^{\forall} B \to C$.

Our claim is that $\mathcal{M}_{\mathcal{Cl}}, x \Vdash (A \vee B) > C$, meaning that for each $\gamma \in N(x)$, if $\gamma \Vdash^{\exists} A \vee B$ then there exists γ' such that:

3. $\gamma' \subseteq \gamma$ and
4. $\gamma' \Vdash^{\exists} A \vee B$ and
5. $\gamma' \Vdash^{\forall} (A \vee B) \to C$.

Assume that for some γ it holds that $\gamma \Vdash^{\exists} A \vee B$, and that $\gamma \Vdash^{\exists} A$ (the case of $\gamma \Vdash^{\exists} B$ is proved symmetrically). Condition 1 guarantees that there exists a γ' such that

6. $\gamma' \subseteq \gamma$ and
7. $\gamma' \Vdash^{\exists} A$ and
8. $\gamma' \Vdash^{\forall} A \to C$.

Condition 6 meets requirement 3, and from 7 we have that $\gamma' \Vdash^{\exists} A \vee B$, meeting requirement 4. If it holds that $\gamma' \Vdash^{\forall} B \to C$, we can use 8 and obtain that $\gamma' \Vdash^{\forall} (A \vee B) \to C$, thus meeting requirement 5 and concluding the proof. Now suppose $\gamma' \nVdash^{\forall} B \to C$. Then, $\gamma' \Vdash^{\exists} B \wedge \neg C$, and $\gamma' \Vdash^{\exists} B$. Applying 2 we obtain that there exists γ'' such that:

9. $\gamma'' \subseteq \gamma'$ and
10. $\gamma'' \Vdash^{\exists} B$ and
11. $\gamma'' \Vdash^{\forall} B \to C$.

In this case, we use γ'' to meet requirements 3, 4 and 5. It holds that:

12. $\gamma'' \subseteq \gamma$, from 6 and 9 (requirement 3 is met);
13. $\gamma'' \Vdash^{\exists} A \vee B$, from 10 (requirement 4 is met);
14. $\gamma'' \Vdash^{\forall} A \to C$, from 8 and 9;
15. $\gamma'' \Vdash^{\forall} (A \vee B) \to C$, from 14 and 11 (requirement 5 is met).

(U_1) $(\neg A > \bot) \to (\neg(\neg A > \bot) > \bot)$. Suppose there is a neighbourhood model with local uniformity that verifies the antecedent: $\mathcal{M}_{\mathcal{U}}, x \Vdash \neg A > \bot$, meaning that for each $\alpha \in N(x)$, it holds that:

1. if $\alpha \Vdash^{\exists} \neg A$,
2. then there exists $\alpha' \subseteq \alpha$ such that $\alpha' \Vdash^{\exists} \neg A$ and $\alpha' \Vdash^{\forall} \neg A \to \bot$.

Condition 2 implies that $\alpha' \Vdash^{\exists} \bot$; thus, 1 cannot hold, and we conclude that:

3. for all $\alpha \in N(x)$, $\alpha \Vdash^{\forall} A$.

We need to prove that $\mathcal{M}_{\mathcal{U}}, x \Vdash \neg(\neg A > \bot) > \bot$, meaning that for all $\alpha \in N(x)$, if $\alpha \Vdash^{\exists} \neg(\neg A > \bot)$, then there exists $\alpha' \subseteq \alpha$ such that $\alpha' \Vdash^{\exists} \neg(\neg A > \bot)$ and $\alpha' \Vdash^{\forall} \neg(\neg A > \bot) \to \bot$.

Following the same reasoning as before, we conclude that this amounts at proving that for all $\alpha \in N(x)$, $\alpha \Vdash^{\exists} \neg A > \bot$. Let us choose an arbitrary $\alpha \in N(x)$, and let $y \in \alpha$. We have to show that $y \Vdash \neg A > \bot$ holds, i.e., that for $\beta \in N(y)$, if $\beta \Vdash^{\exists} \neg A$, then there exists $\beta' \subseteq \beta$ such that $\beta' \Vdash^{\exists} \neg A$ and $\beta' \Vdash^{\forall} \neg A \to \bot$. Again, this amounts at proving the following:

4. for all $\beta \in N(y)$, $\beta \Vdash^{\forall} A$.

Since $y \in \alpha$ and $\alpha \in N(y)$, by local uniformity we conclude that $\bigcup N(x) = \bigcup N(y)$. From this and 3 we conclude that 4 holds, proving the statement. $\square$

3.2 Completeness of $\mathbb{PCL}$

We prove here the completeness of $\mathbb{PCL}$ with respect to neighbourhood semantics (extensions are treated in Subsection 3.3).

Generally speaking, proving completeness for the axiom systems of $\mathbb{PCL}$ and its extensions seems to be quite an arduous task. Burgess [2] was the first to provide a completeness proof for $\mathbb{PCL}$, using preferential models. His proof in the mentioned paper, condensed in a few pages, is quite intricate and not so easy to grasp. In his thesis, Veltman [29] gave a proof of strong completeness of $\mathbb{PCL}$ with respect to preferential semantics. This result is far from elementary. In [5], Halpern and Friedman sketched a completeness proof for $\mathbb{PCL}$, claiming the proof to be similar to Burgess' proof. Moreover, they state that the proof can cover extensions of $\mathbb{PCL}$, but the proof for extensions is postponed to a full paper which never appeared.

More recently, in [8], completeness of the axiomatization of $\mathbb{PCL}$ and its extensions has been proved with respect to classes of preferential models, assuming the Limit assumption.

For Lewis' sphere models, a direct completeness result was given by Lewis in [16]: he proved that the axioms of $\mathbb{V}$ and extensions are sound and complete with respect to sphere models. However, the proof heavily relies on the connective of comparative plausibility, which is definable in $\mathbb{V}$ but not in $\mathbb{PCL}$.

To the best of our knowledge, no completeness result is known for the axioms of $\mathbb{PCL}$ and its extensions with respect to neighbourhood models. The proofs in the rest of this section cover $\mathbb{PCL}$ and all its extensions, except those containing weak centering (and not containing centering). The proofs make use of some notions and lemmas from [8].

We follow the standard strategy: in order to prove completeness of an axiom system $\mathcal{H}_{\mathsf{PK}}$ with respect to a class of models $\mathcal{M}_{\mathcal{K}}$, we define a model $\mathfrak{M}_{\mathcal{K}}$ and we prove that:

1. $\mathfrak{M}_{\mathcal{K}}$ is *canonical*, meaning that for any formula $F \in \mathcal{L}$, $\vdash_{\mathsf{PK}} F$ if and only if F is valid in $\mathfrak{M}_{\mathcal{K}}$;

2. $\mathfrak{M}_{\mathcal{K}} \in \mathcal{M}_{\mathcal{K}}$.

From these two facts the completeness of $\mathcal{H}_{\mathsf{PK}}$ with respect the class $\mathcal{M}_{\mathcal{K}}$ immediately follows. For $\mathbb{PCL}$ the class $\mathcal{M}_{\mathcal{K}}$ will be the class of all neighbourhood models, $\mathcal{M}_{\mathcal{C}l}$; for extensions, $\mathcal{K}$ will be one of $\mathcal{N}$, $\mathcal{T}$, $\mathcal{W}$, $\mathcal{C}, \mathcal{U}$, $\mathcal{A}$, $\mathcal{NU}$, $\mathcal{TU}$, $\mathcal{WU}$, $\mathcal{CU}$, $\mathcal{NA}$, $\mathcal{TA}$, $\mathcal{WA}$, $\mathcal{CA}$, and $\mathcal{M}_{\mathcal{K}}$ will denote the class of models extended with the relevant semantic conditions.

As usual, the model is built by considering *maximal consistent sets* of formulas. We start by recalling standard definitions and properties. The notion of (in-)consistency and subsequent definitions and lemmas on maximal consistent sets are relative to some axiom system $\mathcal{H}_{\mathsf{PK}}$.

Definition 3.2. Given a set of formulas $S \in \mathcal{L}$, we say that S is *inconsistent* if it has a finite subset $\{B_1, \dots, B_n\} \subseteq S$ such that $\vdash_{\mathsf{PK}} (B_1 \wedge \dots \wedge B_n) \to \bot$. We say that S is *consistent* if it is not inconsistent. We say that S is *maximal consistent* if S is consistent and for any formula $A \notin S$, $S \cup \{A\}$ is inconsistent. We denote by $X, Y, Z, \dots$ the maximal consistent sets and by MAX the set of all maximal consistent sets over $\mathcal{L}$.

We assume all standard properties of MAX sets, in particular the following:

Lemma 3.3.

a) *For a set S of formulas, S is consistent if and only if there exists $Z \in \mathsf{MAX}$ such that $S \subseteq Z$.*

b) *For a formula A, $\vdash_{\mathsf{PK}} A$ if and only if for all $Z \in \mathsf{MAX}$, $A \in Z$.*

Proof. *a*) is standard, in particular the direction *only if* is the standard *Lindenbaum Lemma* (see for instance [4, Section 2.6]). *b*) is obtained by *a*) by contraposition and completeness of any $Z \in \mathsf{MAX}$ (either $A \in Z$ or $\neg A \in Z$), since $\not\vdash_{\mathsf{PK}} A$ iff $S = \{\neg A\}$ is consistent. □

We will define the worlds of the canonical model $\mathfrak{M}_{\mathcal{K}}$ in Definition 3.10, as the set $\{(X, A) \mid X \in \mathsf{MAX} \textit{ and } A \in \mathcal{L} \textit{ and } A \in X\}$. Thanks to Lemma 3.3, in order to prove that $\mathfrak{M}_{\mathcal{K}}$ is indeed *canonical*, we will only have to show that for any formula $F \in \mathcal{L}$ and for any world (X, A), it holds that:

$$(\textit{Truth Lemma}) \quad F \in X \text{ if and only if } (X, A) \Vdash F.$$

Canonicity of $\mathfrak{M}_{\mathcal{K}}$ easily follows from the Truth Lemma: A is valid in $\mathfrak{M}_{\mathcal{K}}$ if and only if for all $Z \in \mathsf{MAX}$, $A \in Z$ (by the Truth Lemma and definition of the worlds), if and only if $\vdash_{\mathsf{PK}} A$ (by Lemma 3.3).

Before providing the canonical model construction, we introduce some additional definitions and lemmas.

Definition 3.4. Let $X \in \mathsf{MAX}$. The set of *conditional consequences* of a formula $B \in \mathcal{L}$ at X is defined as: $X^B = \{C \in \mathcal{L} \mid B > C \in X\}$.

Lemma 3.5. *The following hold:*

1. $B \in X^B$;

2. *If* $X^B \subseteq Y$ *and* $B > C \in X$*, then* $C \in Y$*;*

3. $B > C \in X$ *iff for all* Y*,* $X^B \subseteq Y$ *implies* $C \in Y$*.*

Proof. We prove only direction $\Leftarrow$ of statement 3. By hypothesis, there is no $Z \in \mathsf{MAX}$ such that $X^B \cup \{\neg C\} \subseteq Z$. By Lemma 3.3, $X^B \cup \{\neg C\}$ is inconsistent, and there must be some $D_1, \dots, D_n \in X^B$ such that $\vdash_{\mathsf{PK}} (D_1 \wedge \cdots \wedge D_n) \to C$. Thus, by (RCK), $\vdash_{\mathsf{PK}} ((B > D_1) \wedge \cdots \wedge (B > D_n)) \to (B > C)$. Since $(B > D_1), \dots, (B > D_n) \in X$, also $B > C \in X$. □

Definition 3.6. Let $X \in \mathsf{MAX}$, $A, B \in \mathcal{L}$. Define $A \leqslant_X B$ if $(A \vee B) > A \in X$.

Proposition 3.7. *The relation* $\leqslant_X$ *is reflexive and transitive.*

Proof. Reflexivity follows from axiom (ID) and (OR). Transitivity immediately follows from formula 5 of Proposition 2.2. □

Proposition 3.8 (From [8])**.** *If* $A \leqslant_X B$*,* $X^A \subseteq Y$ *and* $B \in Y$*, then* $X^B \subseteq Y$*.*

Proof. Let $B > C \in X$ (thus, $C \in X^B$). Our goal is to show that $C \in Y$. By hypothesis, we know that $(A \vee B) > A \in X$. From formula 7 of Proposition 2.2 it follows that $A > (B \to C) \in X$. Thus, $B \to C \in X^A$ and, by hypothesis $B \to C \in Y$ and $B \in Y$. Thus, $C \in Y$. □

Proposition 3.9. *If* $A \leqslant_X B \leqslant_X C$*,* $X^A \subseteq Y$ *and* $C \in Y$*, then* $X^B \subseteq Y$*.*

Proof. By hypothesis, $(A \vee B) > A \in X$ and $(B \vee C) > B \in X$. By formula 6 of Proposition 2.2, $A > (C \to B) \in X$. Thus, $C \to B \in X^A$, and $C \to B \in Y$. Since $C \in Y$, we have $B \in Y$. Thus, we have that $A \leqslant_X B$, $X^A \subseteq Y$ and $B \in Y$. Applying Proposition 3.8 we obtain $X^B \subseteq Y$. □

We can now proceed with the construction of the canonical model. Before providing the formal definition, let us give some informal explanation about the construction. In standard canonical model constructions, worlds are identified with maximal consistent sets X, Y,...; however, as we will see, we will need a slightly more complex structure.

The most important task in the construction of the canonical model is the definition of the neighbourhood function. We may expect the neighbourhoods of a world X to be determined by the conditionals "holding" in X[7] or, more precisely, by the *antecedents* of conditionals holding in X. Let us call *B-conditional* a conditional whose antecedent is the formula B. Moreover, let us say that, for a world Y, Y *fulfils the B-conditionals of* X if $X^B \subseteq Y$, and, for a neighbourhood β, β *fulfils the B-conditionals of* X if β contains a world fulfilling the B-conditionals of X. Thus, we may define the neighbourhoods of X as containing the worlds fulfilling the B-conditionals of X, along with distinct neighbourhoods containing worlds fulfilling, say, the C-conditionals of X.

For two non-equivalent formulas B and C, it may happen that $X^B \subseteq Y$ and $X^C \not\subseteq Y$. As a consequence, Y will belong to a neighbourhood of X determined

[7]We loosely say that a formula A "holds" in X, meaning that $A \in X$.

by the B-conditionals, but it will not belong to a neighbourhood of X determined by the C-conditionals. The *same* Y has therefore a *different* "status" depending on whether we consider B-conditionals, rather than C-conditionals.

This leads us to "label" worlds with each formula they contain: the idea is that the designated formula plays the role of the antecedent of conditionals potentially fulfilled by that world. Thus, we consider worlds as *pairs*: in this way, world (Y, B) will be different from world (Y, C) (with $B, C \in Y$).

Since worlds in the model are now pairs (Y, B), the neighbourhood function will be defined on such pairs: the formula B in a pair (Y, B) is *not* needed to define the neighbourhoods of world (Y, B)[8], but it is needed to know whether (Y, B) will included or not in some neighbourhood of some other world (X, A). This latter will depend on whether Y fulfils the B-conditionals of X or not.

This definition of worlds as pairs is the same as the one given in [7] (although the notion of model is completely different from [7]); moreover, a somewhat similar idea is adopted in the completeness proof by Friedman and Halpern [5, Theorem 5.2]).

Definition 3.10 below (introducing the *canonical model*) ensures that for distinct maximal consistent sets X, Y and formula $B \in Y$, if Y fulfils the B-conditionals of X, then the system of neighbourhoods associated to X will contain a neighbourhood α with the following features:

1. $(Y, B) \in \alpha$ and it is the *only* world in α at which B holds (i.e. with $B \in Y$);
2. Each world $(Z, C) \in \alpha$ gives rise to a sub-neighbourhood β of α, such that β fulfils the C-conditionals for some C "stronger" than B w.r.t. X (i.e. such that $C \leqslant_X B$).

As we shall see, both features are crucial for the Truth Lemma 3.14: 1. is the key feature used in the second half of the Truth Lemma itself, whereas 2. is the content of Lemma 3.13 and is largely used in the proof.

Definition 3.10. For a propositional atom p, let

- $\mathcal{W} = \{(X, A) \mid X \in \mathsf{MAX} \textit{ and } A \in \mathcal{L} \textit{ and } A \in X\}$;
- $\mathcal{V}(p) = \{(X, A) \in \mathcal{W} \mid p \in X\}$.

For $(X, A), (Y, B) \in \mathcal{W}$, we define a neighbourhood as:

$$\nu^{(X,A)}_{(Y,B)} = \{(Z, C) \in \mathcal{W} \mid X^C \subseteq Z \textit{ and } C \leqslant_X B \textit{ and } B \notin Z\} \cup \{(Y, B)\}$$

Now for any $(X, A) \in \mathcal{W}$, let the neighbourhood function be defined as :

$$\mathcal{N}((X, A)) = \{\nu^{(X,A)}_{(Y,B)} \mid X^B \subseteq Y \textit{ and } B \in \mathcal{L}\}$$

Finally, let the *canonical model* be defined as $\mathfrak{M}_{\mathcal{C}l} = \langle \mathcal{W}, \mathcal{N}, \mathcal{V} \rangle$.

Notation 3.11. Slightly abusing the notation, we write $\mathcal{N}(X, A)$ instead of $\mathcal{N}((X, A))$. Moreover, since the A in $\nu^{(X,A)}_{(Y,B)}$ is not needed[9], we simplify the

[8] Formula B will however be relevant for some extensions of $\mathbb{PCL}$.

[9] For $\mathbb{PCL}$ at least, see previous footnote.

notation to $\nu^X_{(Y,B)}$.

Proposition 3.12. *The canonical model $\mathfrak{M}_{\mathcal{C}l}$ is a neighbourhood model.*

Proof. It suffices to verify that that non-emptiness holds; since for all $(Y, B) \in \mathcal{W}$ it holds that $(Y, B) \in \nu^X_{(Y,B)}$, the property immediately follows. $\square$

Lemma 3.13. *If $\nu^X_{(Y,B)} \in \mathcal{N}(X, A)$ and $(U, D) \in \nu^X_{(Y,B)}$, then $\nu^X_{(U,D)} \subseteq \nu^X_{(Y,B)}$.*

Proof. We prove the non-trivial case in which $(U, D) \neq (Y, B)$. Let $(V, E) \in \nu^X_{(U,D)}$ and $(V, E) \neq (U, D)$. We will show that $(V, E) \in \nu^X_{(Y,B)}$. By definition, this amounts to show that:

a. $X^E \subseteq V$ and
b. $E \leqslant_X B$ and
c. $B \notin V$.

Since $(V, E) \in \nu^X_{(U,D)}$, by definition we have that:

a′. $X^E \subseteq V$ and
b′. $E \leqslant_X D$ and
c′. $D \notin V$.

Thanks to a', requirement a is met. Since $(U, D) \in \nu^X_{(Y,B)}$, it follows by definition that:

a″. $X^D \subseteq U$ and
b″. $D \leqslant_X B$ and
c″. $B \notin U$.

From b' and b'' it follows by transitivity of $\leqslant_X$ (Proposition 3.7) that $E \leqslant_X B$, and requirement b is met. It remains to prove that $B \notin V$. For the sake of contradiction, suppose that $B \in V$. From $B \in V$ together with $E \leqslant_X D \leqslant_X B$ (b' + b'') and $X^E \subseteq V$ (a') it follows by Proposition 3.9 that $X^D \subseteq V$. By Lemma 3.5 we have that $D \in X^D$; from $D \in X^D$ and $X^D \subseteq V$ it follows that $D \in V$. This contradicts c'. Therefore $B \notin V$, as required. $\square$

We are now ready to prove the Truth Lemma.

Lemma 3.14 (Truth Lemma)**.** *For all $F \in \mathcal{L}$, $(X, A) \in \mathcal{W}$ the following statements are equivalent:*

- *$F \in X$;*
- *$\mathfrak{M}_{\mathcal{C}l}, (X, A) \Vdash F$.*

Proof. The proof proceeds by induction on the structure of F. We show only the case of $F \equiv G > H$, assuming the following inductive hypothesis:
for all $(X, A) \in \mathcal{W}$, $\mathfrak{M}_{\mathcal{C}l}, (X, A) \Vdash J$ iff $J \in X$, with $J \in \{G, H\}$. We shall prove the equivalence of the following statements:

1. $G > H \in X$;
2. For all $\alpha \in \mathcal{N}(X, A)$, if $\alpha \Vdash^{\exists} G$ then there exists $\beta \in \mathcal{N}(X, A)$ with $\beta \subseteq \alpha$, $\beta \Vdash^{\exists} G$ and $\beta \Vdash^{\forall} G \to H$.

[**1** $\Rightarrow$ **2**] Assume 1, and suppose that $\alpha \in \mathcal{N}(X, A)$ and $\alpha \Vdash^{\exists} G$, for $\alpha = \nu^X_{(Y,B)}$. We must show that there exists a $\beta \in \mathcal{N}(X, A)$ such that $\beta \subseteq \alpha$, $\beta \Vdash^{\exists} G$ and $\beta \Vdash^{\forall} G \to H$.

We distinguish two cases, depending on whether $B \leqslant_X G$ holds or not. Suppose it holds; then, we show that we can take $\beta = \alpha = \nu^X_{(Y,B)}$. Given the hypothesis we only have to prove that $\alpha \Vdash^{\forall} G \to H$. To this aim let $(U, D) \in \nu^X_{(Y,B)}$ and $G \in U$. From $(U, D) \in \nu^X_{(Y,B)}$ it follows that $X^D \subseteq U$ and $D \leqslant_X B$. Since $B \leqslant_X G$, by transitivity of $\leqslant_X$ we obtain $D \leqslant_X G$. Therefore we have: $G \in U$, $X^D \subseteq U$ and $B \leqslant_X G$,so that by Proposition 3.8 we obtain $X^G \subseteq U$. Since $G > H \in X$ we have $H \in X^G$, therefore $H \in U$.

Now suppose that $B \leqslant_X G$ does not hold. Therefore $\neg((B \vee G) > B) \in X$. Thus, $X^{B\vee G} \cup \{\neg B\}$ is consistent, so that (by Lemma 3.3) there exists some $Z \in \mathsf{MAX}$ such that $X^{B\vee G} \cup \{\neg B\} \subseteq Z$ (whence $G \in Z$). Let us consider the world $(Z, B \vee G)$. The following hold:

a. $X^{B\vee G} \subseteq Z$;
b. $(B \vee G) \leqslant_X B$;
c. $B \notin Z$.

Statements a and c hold by construction; as for statement b, it means by definition that $(B \vee G \vee B) > (B \vee G) \in X$, which immediately follows from $(B \vee G) > (B \vee G) \in X$, since $B \vee G \vee B = B \vee G$. Thus, from a, b and c we obtain by Definition 3.10 that $(Z, B \vee G) \in \nu^X_{(Y,B)}$. We show that we can take $\beta = \nu^X_{(Z,B\vee G)}$: since $X^{B\vee G} \subseteq Z$, we have $\nu^X_{(Z,B\vee G)} \in \mathcal{N}(X, A)$; since $(Z, B \vee G) \in \nu^X_{(Y,B)}$, by Lemma 3.13 we have $\nu^X_{(Z,B\vee G)} \subseteq \nu^X_{(Y,B)}$; since $G \in Z$, we immediately have $\nu^X_{(Z,B\vee G)} \Vdash^{\exists} G$. We still have to prove that $\nu^X_{(Z,B\vee G)} \Vdash^{\forall} G \to H$. To this purpose suppose $(U, D) \in \nu^X_{(Z,B\vee G)}$ and $G \in U$, we must show that $H \in U$. We know that $X^G \subseteq U$, $G \in U$, and $D \leqslant_X B \vee G \leqslant_X G$, thus we may apply proposition by Proposition 3.8, and obtain $X^G \subseteq U$ from which we conclude $H \in U$.

[**2** $\Rightarrow$ **1**] Assume 2. We show that for all $Z \in \mathsf{MAX}$, if $X^G \subseteq Z$, then $H \in Z$. By Lemma 3.5, this is equivalent to $G > H \in X$.

To this aim, suppose that $X^G \subseteq Z$, for some Z. Then, $(Z, G) \in \mathcal{W}$. Let us consider the neighbourhood $\nu^X_{(Z,G)} = \alpha$: by construction this neighbourhood belongs to $\mathcal{N}(X, A)$ and thus, by hypothesis, $\nu^X_{(Z,G)} \Vdash^{\exists} G$. Let us now assume that there exists some neighbourhood $\beta \in \mathcal{N}(X, A)$ such that $\beta \subseteq \alpha$, $\beta \Vdash^{\exists} G$ and $\beta \Vdash^{\forall} G \to H$. It easy to see that it must be $\beta = \alpha = \nu^X_{(Z,G)}$, since by Definition 3.10 the only world that satisfies G in the neighbourhood $\nu^X_{(Z,G)}$ is (Z, G) itself, because for all $(U, D) \in \nu^X_{(Z,G)}$ if $(U, D) \neq (Z, G)$ then $G \notin U$.

Thus, from $\nu^X_{(Z,G)} \Vdash^\forall G \rightarrow H$, $(Z,G) \in \nu^X_{(Z,G)}$ and $G \in Z$ it immediately follows that $H \in Z$. $\square$

Since for any set of formulas X and formula $A \in X$, by definition $(X, A) \in \mathcal{W}$ iff $X \in MAX$, by the previous Lemma and Lemma 3.3 we immediately obtain:

Theorem 3.15 (Completeness)**.** *For $F \in \mathcal{L}$, if $\models_{\mathcal{C}l} F$ then $\vdash_{\mathsf{PCL}} F$.*

3.3 Completeness for extensions of $\mathbb{PCL}$

Our aim is to extend the completeness proof to the whole family of all preferential logics. We are able to extend the proof to all extensions of $\mathbb{PCL}$, except for the systems containing weak centering (and not containing centering). To obtain a proof for a logic featuring more than one semantic condition, it suffices to combine the proof strategies for each case.

Unless otherwise specified, all notions refer to the canonical model for $\mathbb{PCL}$ defined in the previous section. In some cases, the canonical model needs to be modified to account for specific conditions. The following proposition (whose proof is obvious) will used for the cases of absoluteness and uniformity.

Proposition 3.16. *For every $(X, A) \in \mathcal{W}$, it holds:*

$$\bigcup \mathcal{N}(X, A) = \{(Z, C) \in \mathcal{W} \mid X^C \subseteq Z\}.$$

Normality

The canonical model for the case of normality, $\mathfrak{M}_{\mathcal{N}}$, is defined in the same way as $\mathfrak{M}_{\mathcal{C}l}$. We show that in presence of axiom (N), $\mathfrak{M}_{\mathcal{N}}$ satisfies the condition of normality:

For all $(X, A) \in \mathcal{W}$, it holds that $\mathcal{N}(X, A) \neq \emptyset$.

By Axiom (N), we have that for all $(X, A) \in \mathcal{W}$, it holds that $\neg(\top > \bot) \in X$. Thus, $X^\top$ is consistent and by Lemma 3.3 there is $Z \in \mathsf{MAX}$ such that $X^\top \subseteq Z$. As a consequence, $(Z, \top) \in \mathcal{W}$, and $\nu^X_{(Z,\top)} \in \mathcal{N}(X, A)$, whence $\mathcal{N}(X, A) \neq \emptyset$.

Absoluteness

The canonical model for the case of normality, $\mathfrak{M}_{\mathcal{A}}$, is defined in the same way as $\mathfrak{M}_{\mathcal{C}l}$. We show that in presence of axioms $(A_1), (A_2)$, the canonical model $\mathfrak{M}_{\mathcal{A}}$ satisfies the condition of local absoluteness:

If $(Z, C) \in \bigcup \mathcal{N}(X, A)$, then $\mathcal{N}(X, A) = \mathcal{N}(Z, C)$.

We first prove that $a)$ for any formula $B \in \mathcal{L}$, $X^B = Z^B$. To this aim, let $G \in X^B$; then $B > G \in X$. By axiom (A_1), $C > (B > G) \in X$, and $B > G \in X^C$. Since $(Z, C) \in \bigcup \mathcal{N}(X, A)$, it holds that $X^C \subseteq Z$; from this follows that $B > G \in Z$, and thus $G \in Z^B$. Conversely, suppose $G \notin X^B$.

Then $\neg(B > G) \in X$; by (A$_2$) $C > \neg(B > G) \in X$, and $\neg(B > G) \in X^C$. Again, since $X^C \subseteq Z$ we have $\neg(B > G) \in Z$, and thus $G \notin Z^B$.

We will show that *b*) for all formulas $D, E \in \mathcal{L}$, it holds $D \leqslant_X E$ if and only if $D \leqslant_Z E$. This is because from $D \leqslant_Z E$ follows that $(D \vee E) > D \in Z$, and by proceeding similarly as in *a*) we obtain that $(D \vee E) > D \in X$ if and only if $(D \vee E) > D \in Z$.

From *a*) it immediately follows that for any (Y, B), $X^B \subseteq Y$ if and only if $Z^B \subseteq Y$. Then, by *b*) we have $\nu^X_{(Y,B)} = \nu^Z_{(Y,B)}$, whence by *a*) we obtain $\mathcal{N}(X, A) = \mathcal{N}(Z, C)$.

Total Reflexivity

In this case we need to modify the construction of the canonical model.

Definition 3.17. The *universe* of (X, A) is the set:

$$\mathsf{Univ}(X, A) = \{(Y, B) \in \mathcal{W} \mid \textit{ for all } G \in \mathcal{L},\ G > \bot \in X \textit{ implies } \neg G \in Y\}.$$

The canonical model $\mathfrak{M}_{\mathcal{T}} = \langle \mathcal{W}^{\mathsf{u}}, \mathcal{N}^{\mathsf{u}}, \mathcal{V}^{\mathsf{u}} \rangle$ is defined by stipulating $\mathcal{W}^{\mathsf{u}} = \mathcal{W}$, $\mathcal{V}^{\mathsf{u}} = \mathcal{V}$, and

$$\mathcal{N}^{\mathsf{u}}(X, A) = \mathcal{N}(X, A) \cup \{\mathsf{Univ}(X, A)\}.$$

where $\mathcal{N}(X, A)$ is the same as in Definition 3.10.

Lemma 3.18. *For any* $(X, A), (Y, B) \in \mathcal{W}$*, it holds that* $\nu^X_{(Y,B)} \subseteq \mathsf{Univ}(X, A)$.

Proof. Assume that some $(Z, C) \in \nu^X_{(Y,B)}$. We have to prove that for all $G \in \mathcal{L}$, if $G > \bot \in X$ then $\neg G \in Z$, and this immediately follows from (MOD) (2 of Proposition 2.2) and $X^C \subseteq Z$. □

We show that in presence of axiom (T), the canonical model $\mathfrak{M}_{\mathcal{T}}$ satisfies the condition of total reflexivity, that is:

$$\text{If } (X, A) \in \mathcal{W}^{\mathsf{u}}, \text{ there exists } \alpha \in \mathcal{N}^{\mathsf{u}}(X, A) \text{ such that } (X, A) \in \alpha.$$

It is immediate to verify that the condition holds: because of axiom (T), we have that $(X, A) \in \mathsf{Univ}(X, A)$.

Since we have modified the definition of the canonical model, we have to verify that the Truth Lemma still holds. To this aim, we need to add one case in the direction $[\mathbf{1} \Rightarrow \mathbf{2}]$ of the proof, that is, if $G > H \in X$, then $\mathfrak{M}_{\mathcal{T}}, (X, A) \Vdash G > H$. Assume that $G > H \in X$ and that for some $\alpha \in \mathcal{N}^{\mathsf{u}}(X, A)$ it holds $\alpha \Vdash^{\exists} A$. If $\alpha \in \mathcal{N}(X, A)$ the proof proceeds as in Lemma 3.14. Let now $\alpha = \mathsf{Univ}(X, A)$ and suppose for some $(Z, C) \in \mathsf{Univ}(X, A)$ it holds that $(Z, C) \Vdash G$, whence $G \in Z$. We show that there must exist an $(U, D) \in \bigcup \mathcal{N}(X, A)$ such that $(U, D) \Vdash G$. If this were not the case, we would get that for all $(U, D) \in \bigcup \mathcal{N}(X, A)$, $G \notin U$. But this entails that X^G is inconsistent; and thus $G > \bot \in X$, against the hypothesis that $(Z, C) \Vdash G$ and $(Z, C) \in \mathsf{Univ}(X, A)$. Thus there is a $(U, D) \in \bigcup \mathcal{N}(X, A)$ such that $G \in U$. We take $\alpha' = \nu^X_{(U,D)}$. Observe that $\alpha' \subseteq \alpha = \mathsf{Univ}(X, A)$. We can proceed as in proof of Lemma 3.14 by finding a $\beta \in \mathcal{N}(X, A)$ with $\beta \subseteq \alpha'$ fulfilling the required conditions.

Uniformity

We take the same model construction as for total reflexivity: $\mathfrak{M}_{\mathcal{U}} = \mathfrak{M}_{\mathcal{T}}$. Thus, we only need to show that in presence of axioms (U_1) and (U_2) $\mathfrak{M}_{\mathcal{U}}$ satisfies the condition of local uniformity, that is, for any $(X, A), (Y, B) \in \mathcal{W}^{\mathsf{u}}$:

$$\text{If } (Y, B) \in \bigcup \mathcal{N}^{\mathsf{u}}(X, A), \text{ then } \bigcup \mathcal{N}^{\mathsf{u}}(X, A) = \bigcup \mathcal{N}^{\mathsf{u}}(Y, B).$$

To this aim, first observe that

$$\bigcup \mathcal{N}^{\mathsf{u}}(X, A) = \mathsf{Univ}(X, A)$$

Suppose now $(Y, B) \in \bigcup \mathcal{N}^{\mathsf{u}}(X, A) = \mathsf{Univ}(X, A)$. We show that $G > \bot \in X$ if and only if $G > \bot \in Y$. Let $G > \bot \in X$. Then by axiom (U_1) it follows that $\neg(G > \bot) > \bot \in X$. Since $(Y, B) \in \mathsf{Univ}(X, A)$ we have $\neg\neg(G > \bot) \in Y$, that is $G > \bot \in Y$. Conversely, suppose that $G > \bot \notin X$, i.e., $\neg(G > \bot) \in X$. By axiom (U_2) we have that $(G > \bot) > \bot \in X$, and since $(Y, B) \in \mathsf{Univ}(X, A)$, we get $\neg(G > \bot) \in Y$, whence $G > \bot \notin Y$.
From the fact that $G > \bot \in X$ if and only if $G > \bot \in Y$ we obtain that for all $(Z, C) \in \mathcal{W}^{\mathsf{u}}$, $(Z, C) \in \mathsf{Univ}(X, A)$ if and only if $(Z, C) \in \mathsf{Univ}(Y, B)$, which means $\bigcup \mathcal{N}^{\mathsf{u}}(X, A) = \bigcup \mathcal{N}^{\mathsf{u}}(Y, B)$.

Centering

We modify the canonical model construction as follows.

Definition 3.19. The canonical model $\mathfrak{M}_{\mathcal{C}} = \langle \mathcal{W}^{\mathsf{c}}, \mathcal{N}^{\mathsf{c}}, \mathcal{V}^{\mathsf{c}} \rangle$ is defined by stipulating $\mathcal{W}^{\mathsf{c}} = \mathcal{W}$ and $\mathcal{V}^{\mathsf{c}} = \mathcal{V}$. For the neighbourhood function, let us define $(X, A), (Y, B) \in \mathcal{W}^{\mathsf{c}}$:

$$\mu_{(Y,B)}^{(X,A)} = \nu_{(Y,B)}^{X} \cup \{(X, A)\}.$$

Observe that here the formula A in (X, A) is relevant. Then, for any $(X, A) \in \mathcal{W}$, $\mathcal{N}^{\mathsf{c}}(X, A) = \{\mu_{(Y,B)}^{(X,A)} \mid X^B \subseteq Y\}$.

We now show that in presence of axioms W and C, the canonical model $\mathfrak{M}_C$ satisfies the condition of centering:

a) For every world (X, A) and every $\alpha \in \mathcal{N}^{\mathsf{c}}(X, A)$, $(X, A) \in \alpha$;

b) $\{(X, A)\} \in \mathcal{N}^{\mathsf{c}}(X, A)$.

Condition *a)* holds by definition. As for *b)*, first observe that for any (X, A) it holds by (W) that $X^A \subseteq X$, so that $\mu_{(X,A)}^{(X,A)} \in \mathcal{N}^{\mathsf{c}}(X, A)$. We now show that $\mu_{(X,A)}^{(X,A)} = \{(X, A)\}$. To this aim, we prove that there is no world $(Y, B) \in \mu_{(X,A)}^{(X,A)}$ such that $(Y, B) \neq (X, A)$. For the sake of contradiction, suppose such a world exists. It follows that $A \notin Y$ and $B \leqslant_X A$, which means that $(A \vee B) > B \in X$. Thus, by axiom (W), $(A \vee B) \to B \in X$. Since by definition $A \in X$, we have $B \in X$. By axiom (C) it follows that also $B > A \in X$. Thus, $A \in X^B$; and since $X^B \subseteq Y$ we have $A \in Y$, which contradicts with the assumption $A \notin Y$.

Since we have modified the canonical model, we have to verify that the Truth Lemma continues to hold. For the direction [**1** $\Rightarrow$ **2**], suppose that $G > H \in X$ and that for $\alpha \in \mathcal{N}^{\mathsf{c}}(X, A)$ it holds that $\alpha \Vdash^{\exists} G$. We can proceed as in the proof of Lemma 3.14, finding a suitable $\beta \in \mathcal{N}^{\mathsf{c}}(X, A)$. The fact that (X, A) belongs to every neighbourhood in $\mathcal{N}^{\mathsf{c}}(X, A)$, and also to β, does not compromise the assertion that $\beta \Vdash^{\forall} G \to H$, since from the hypothesis $G > H \in X$ follows by (W) that $G \to H \in X$.

For the direction [**2** $\Rightarrow$ **1**], assume 2. We distinguish two cases:

i. $G \notin X$;
ii. $G \in X$.

In case i, we proceed as in the proof of Lemma 3.14, by proving that for all $Z \in \mathsf{MAX}$, if $X^G \subseteq Z$, then $H \in Z$. To this aim, let us consider $\alpha = \mu^{(X,A)}_{(Z,G)} = \nu^X_{(Z,G)} \cup \{(X, A)\} \in \mathcal{N}^{\mathsf{c}}(X, A)$. By hypothesis, there exists a neighbourhood $\beta \in \mathcal{N}^{\mathsf{c}}(X, A)$ such that $\beta \subseteq \alpha$, $\beta \Vdash^{\exists} G$ and $\beta \Vdash^{\forall} G \to H$. Since $G \notin X$, it must be that $\beta = \mu^{(X,A)}_{(Z,G)}$, whence $(Z, G) \in \beta$ follows.

In case ii, let us consider $\alpha = \mu^{(X,A)}_{(X,A)} \in \mathcal{N}^{\mathsf{c}}(X, A)$. By hypothesis, there exists a neighbourhood $\beta \in \mathcal{N}^{\mathsf{c}}(X, A)$ such that $\beta \subseteq \mu^{(X,A)}_{(X,A)}$, $\beta \Vdash^{\exists} G$ and $\beta \Vdash^{\forall} G \to H$. However, since $\mu^{(X,A)}_{(X,A)} = \{(X, A)\}$, it must be $\beta = \{(X, A)\}$. Thus, since $G \to H \in X$ and $G \in X$, we obtain $H \in X$. By axiom (C), we finally obtain $G > H \in X$.

Theorem 3.20 (Completeness for extensions)**.** *Let* $\mathbb{K}$ *be one of* $\mathbb{PN}$*,* $\mathbb{PT}$*,* $\mathbb{PC}$*,* $\mathbb{PU}$*,* $\mathbb{PNU}$*,* $\mathbb{PTU}$*,* $\mathbb{PCU}$*,* $\mathbb{PNA}$*,* $\mathbb{PTA}$*,* $\mathbb{PCA}$*. For* $F \in \mathcal{L}$*, if* $\vDash_{\mathcal{K}} F$*, then* $\vdash_{\mathsf{PK}} F$*.*

4 A family of labelled sequent calculi

In this section we shall introduce a family of labelled sequent calculi for $\mathbb{PCL}$ and its extensions. The calculus for $\mathbb{PCL}$ will be called **G3P.CL**. Calculi for extensions will be denoted by **G3P.CL** with the letters corresponding to the names of the logics added as a superscript: thus, for example, $\mathbf{G3P.CL^{N}}$ and $\mathbf{G3P.CL^{TU}}$ are the calculi for $\mathbb{PCLN}$ and $\mathbb{PCLTU}$, respectively. We use $\mathbf{G3P.CL^{*}}$ to denote any calculus of the family.

The definition of labelled sequent calculi $\mathbf{G3P.CL^{*}}$ follows the well-established methodology of enriching the language of the calculus by means of labels, thus importing the semantic information of neighbourhood models into the proof system[10]. For this reason, it is useful to recall the truth condition for the conditional operator in neighbourhood models:

[10] Refer to [19] for the general methodology in Kripke models and to [20] for the general methodology in neighbourhood semantics.

$$x \Vdash A > B \quad \textit{iff} \quad \begin{array}{l} \text{for all } \alpha \in N(x), \text{ if } \alpha \Vdash^{\exists} A, \\ \text{there exists } \beta \in N(x) \text{ with } \beta \subseteq \alpha \text{ and} \\ \text{such that } \beta \Vdash^{\exists} A \text{ and } \beta \Vdash^{\forall} A \to B. \end{array} \qquad (*)$$

We enrich the language $\mathcal{L}$ as follows.

Definition 4.1. Let $x, y, z, \dots$ be variables for worlds in a neighbourhood model, and $a, b, c, \dots$ variables for neighbourhoods. *Relational atoms* are the following expressions:

- $a \in N(x)$, "neighbourhood a belongs to the family of neighbourhoods associated to x";
- $x \in a$, "world x belongs to neighbourhood a";
- $a \subseteq b$, "neighbourhood a is included into neighbourhood b".

Labelled formulas are defined as follows. Relational atoms are labelled formulas and, for $A \in \mathcal{L}$, the following are labelled formulas:

- $x : A$, "formula A is true at world x";
- $a \Vdash^{\exists} A$, "A is true at some world belonging to neighbourhood a";
- $a \Vdash^{\forall} A$, "A is true at all worlds belonging to neighbourhood a";
- $x \Vdash_a A|B$, "there exists a neighbourhood $b \in N(x)$ such that $b \subseteq a$, $b \Vdash^{\exists} A$, and $b \Vdash^{\forall} A \to B$".

We use $\{x\}$ to denote a neighbourhood consisting of exactly one world.

Relational atoms and labelled formulas are defined in correspondence with semantic notions. Relational atoms describe the structure of the neighbourhood model, whereas labelled formulas are defined in correspondence with the forcing relations at a world ($x \Vdash A$) and at a neighbourhood ($\alpha \Vdash^{\exists} A$ and $\alpha \Vdash^{\forall} A$). Formula $x \Vdash_a A|B$ introduces a semantic condition corresponding to the consequent of the right-hand side of $(*)$. The reason for the introduction of this formula is that $(*)$ is too rich to be expressed by a single rule. Thus we need to break $(*)$ into two smaller conditions, one (the antecedent) covered by $a \Vdash^{\exists} A$ and the other (the consequent) covered by $x \Vdash_a A|B$.

Definition 4.2. Sequents of **G3P.CL*** are expressions $\Gamma \Rightarrow \Delta$ where Γ and Δ are finite multisets of relational atoms and labelled formulas, and relational atoms may occur only in Γ.

Figure 4 contains the rules for $\mathbb{PCL}$, whereas Figure 5 shows the rules for extensions of $\mathbb{PCL}$. Rules for inclusion and all the rules for extensions are *structural rules*, meaning that they only involve relational atoms. We write $(\mathsf{a}!)$, resp. $(\mathsf{x}!)$, as a side condition expressing the requirement that label a, resp. x, should not occur in the conclusion of a rule. Propositional rules are standard. Rules for local forcing make explicit the meaning of the forcing relations $\Vdash^{\forall}$ and $\Vdash^{\exists}$. Rules for the conditional are defined on the basis of the truth condition for $>$ in neighbourhood models.

Initial sequents

$$\frac{}{x : p, \Gamma \Rightarrow \Delta, x : p}\ \mathsf{init} \qquad\qquad \frac{}{x : \bot, \Gamma \Rightarrow \Delta}\ \bot_{\mathsf{L}}$$

Rules for local forcing

$$\frac{x : A, x \in a, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}{x \in a, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}\ \mathsf{L} \Vdash^{\forall} \qquad\qquad \frac{x \in a, \Gamma \Rightarrow \Delta, x : A}{\Gamma \Rightarrow \Delta, a \Vdash^{\forall} A}\ \mathsf{R} \Vdash^{\forall}\ (\mathsf{x!})$$

$$\frac{x \in a, x : A, \Gamma \Rightarrow \Delta}{a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta}\ \mathsf{L} \Vdash^{\exists}\ (\mathsf{x!}) \qquad\qquad \frac{x \in a, \Gamma \Rightarrow \Delta, x : A, a \Vdash^{\exists} A}{x \in a, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A}\ \mathsf{R} \Vdash^{\exists}$$

Propositional rules

$$\frac{x : A, x : B, \Gamma \Rightarrow \Delta}{x : A \wedge B, \Gamma \Rightarrow \Delta}\ \mathsf{L}\wedge \qquad\qquad \frac{\Gamma \Rightarrow \Delta, x : A \quad \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \wedge B}\ \mathsf{R}\wedge$$

$$\frac{x : A, \Gamma \Rightarrow \Delta \quad x : B, \Gamma \Rightarrow \Delta}{x : A \vee B, \Gamma \Rightarrow \Delta}\ \mathsf{L}\vee \qquad\qquad \frac{\Gamma \Rightarrow \Delta, x : A, x : B}{\Gamma \Rightarrow \Delta, x : A \vee B}\ \mathsf{R}\vee$$

$$\frac{\Gamma \Rightarrow \Delta, x : A \quad x : B, \Gamma \Rightarrow \Delta}{x : A \to B, \Gamma \Rightarrow \Delta}\ \mathsf{L}\to \qquad\qquad \frac{x : A, \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \to B}\ \mathsf{R}\to$$

Rules for the conditional

$$\frac{a \in N(x), a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, x \Vdash_a A|B}{\Gamma \Rightarrow \Delta, x : A > B}\ \mathsf{R}>\ (\mathsf{a!})$$

$$\frac{a \in N(x), x : A > B, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A \quad x \Vdash_a A|B, a \in N(x), x : A > B, \Gamma \Rightarrow \Delta}{a \in N(x), x : A > B, \Gamma \Rightarrow \Delta}\ \mathsf{L}>$$

$$\frac{c \in N(x), c \subseteq a, \Gamma \Rightarrow \Delta, x \Vdash_a A|B, c \Vdash^{\exists} A \quad c \in N(x), c \subseteq a, \Gamma \Rightarrow \Delta, x \Vdash_a A|B, c \Vdash^{\forall} A \to B}{c \in N(x), c \subseteq a, \Gamma \Rightarrow \Delta, x \Vdash_a A|B}\ \mathsf{R}|$$

$$\frac{c \in N(x), c \subseteq a, c \Vdash^{\exists} A, c \Vdash^{\forall} A \to B, \Gamma \Rightarrow \Delta}{x \Vdash_a A|B, \Gamma \Rightarrow \Delta}\ \mathsf{L}|\ (\mathsf{c!})$$

Rules for inclusion

$$\frac{a \subseteq a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathsf{ref} \qquad \frac{c \subseteq a, c \subseteq b, b \subseteq a, \Gamma \Rightarrow \Delta}{c \subseteq b, b \subseteq a, \Gamma \Rightarrow \Delta}\ \mathsf{tr} \qquad \frac{x \in a, a \subseteq b, x \in b, \Gamma \Rightarrow \Delta}{x \in a, a \subseteq b, \Gamma \Rightarrow \Delta}\ \mathsf{L}\subseteq$$

Figure 4: Sequent calculus **G3P.CL**

Each rule of Figure 5 is defined in correspondence with the frame conditions on extensions of $\mathbb{PCL}$. For total reflexivity and weak centering, the frame condition can be formalized by means of a single rule. Rule $\mathsf{0}$ stands for the requirement of non-emptiness in the model, and it is added to capture the condition of normality, along with rule N[11].

Centering requires four rules: Rule C ensures the condition by introducing formulas with neighbourhood label $\{x\}$ (the singleton). Rule single ensures that the singleton contains at least one element, and rules repl_1 and repl_2 that it contains at most one element: if there is another element $y \in \{x\}$, then the properties holding for x hold also for y (i.e. x and y are the same element).

[11] The $\mathsf{0}$ rule needs not to be added to the calculus **G3P.CL**: the rules of this calculus always introduce non-empty neighbourhoods, and the system is shown to be complete with respect to the axioms of $\mathbb{PCL}$ in the next Section (Theorem 5.9). However, the rule is needed to express the condition of normality: the new neighbourhood introduced by rule N could be empty.

Rules for extensions

$$\frac{a \in N(x), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathsf{N}\ (\mathsf{a!}) \qquad \frac{y \in a, a \in N(x), \Gamma \Rightarrow \Delta}{a \in N(x), \Gamma \Rightarrow \Delta}\ \mathsf{0}\ (\mathsf{y!})(\star)$$

$$\frac{x \in a, a \in N(x), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathsf{T}\ (\mathsf{a!}) \qquad \frac{x \in a, a \in N(x), \Gamma \Rightarrow \Delta}{a \in N(x), \Gamma \Rightarrow \Delta}\ \mathsf{W}$$

$$\frac{x \in \{x\}, \{x\} \in N(x), \Gamma \Rightarrow \Delta}{\{x\} \in N(x), \Gamma \Rightarrow \Delta}\ \mathsf{single} \qquad \frac{\{x\} \in N(x), \{x\} \subseteq a, a \in N(x), \Gamma \Rightarrow \Delta}{a \in N(x), \Gamma \Rightarrow \Delta}\ \mathsf{C}$$

$$\frac{y \in \{x\}, At(x), At(y), \Gamma \Rightarrow \Delta}{y \in \{x\}, At(x), \Gamma \Rightarrow \Delta}\ \mathsf{repl}_1\ (*) \qquad \frac{y \in \{x\}, At(x), At(y), \Gamma \Rightarrow \Delta}{y \in \{x\}, At(y), \Gamma \Rightarrow \Delta}\ \mathsf{repl}_2\ (*)$$

$$\frac{z \in c, c \in N(x), a \in N(x), y \in a, b \in N(y), z \in b, \Gamma \Rightarrow \Delta}{a \in N(x), y \in a, b \in N(y), z \in b, \Gamma \Rightarrow \Delta}\ \mathsf{U}_1\ (\mathsf{c!})$$

$$\frac{z \in c, c \in N(y), a \in N(x), y \in a, b \in N(y), z \in b, \Gamma \Rightarrow \Delta}{a \in N(x), y \in a, b \in N(x), z \in b, \Gamma \Rightarrow \Delta}\ \mathsf{U}_2\ (\mathsf{c!})$$

$$\frac{b \in N(y), a \in N(x), y \in a, b \in N(x), \Gamma \Rightarrow \Delta}{a \in N(x), y \in a, b \in N(x), \Gamma \Rightarrow \Delta}\ \mathsf{A}_1 \qquad \frac{b \in N(x), a \in N(x), y \in a, b \in N(y), \Gamma \Rightarrow \Delta}{a \in N(x), y \in a, b \in N(y), \Gamma \Rightarrow \Delta}\ \mathsf{A}_2$$

Rules obtained by closure conditions

$$\frac{z \in c, c \in N(x), a \in N(x), x \in a, z \in a, \Gamma \Rightarrow \Delta}{a \in N(x), x \in a, z \in a, \Gamma \Rightarrow \Delta}\ \mathsf{U}_1{}^*\ (\mathsf{c!}) \qquad \frac{a \in N(y), a \in N(x), y \in a, \Gamma \Rightarrow \Delta}{a \in N(x), y \in a, \Gamma \Rightarrow \Delta}\ \mathsf{A}_1{}^*$$

$$\frac{y \in c, c \in N(x), a \in N(x), y \in a, a \in N(y), \Gamma \Rightarrow \Delta}{a \in N(x), y \in a, a \in N(y), \Gamma \Rightarrow \Delta}\ \mathsf{U}_1{}^{**}\ (\mathsf{c!})$$

$$\frac{y \in c, c \in N(y), a \in N(x), y \in a, a \in N(y), \Gamma \Rightarrow \Delta}{a \in N(x), y \in a, \Gamma \Rightarrow \Delta}\ \mathsf{U}_2{}^{**}$$

$(*)$ $At(x) := x : p, x \in a, a \in N(x), x \in \{z\}$, for p atomic formula.

$\mathbf{G3P.N} = \mathbf{G3P.CL} + \mathsf{N} + \mathsf{0}$; $\mathbf{G3P.T} = \mathbf{G3P.N} + \mathsf{T}$; $\mathbf{G3P.W} = \mathbf{G3P.T} + \mathsf{W}$;
$\mathbf{G3P.C} = \mathbf{G3P.W} + \mathsf{C} + \mathsf{single} + \mathsf{repl}_1 + \mathsf{repl}_2$;
$\mathbf{G3P.U} = \mathbf{G3P.CL} + \mathsf{U}_1 + \mathsf{U}_2$; $\mathbf{G3P.NU/TU/WU/CU} = \mathbf{G3P.N/T/W/C}$;
$\mathbf{G3P.A} = \mathbf{G3P.CL} + \mathsf{A}_1 + \mathsf{A}_2$; $\mathbf{G3P.NA/TA/WA/CA} = \mathbf{G3P.N/T/W/C} + \mathsf{A}_1 + \mathsf{A}_2$.

Figure 5: Sequent calculi for extensions of **G3P.CL**

Similarly, extensions with uniformity and absoluteness are defined by adding multiple rules. Rules U_1 and U_2 encode the semantic condition of uniformity. In order to avoid the symbol $\bigcup$ in the sequent language, the rules translate the following two conditions which, taken together, are equivalent to uniformity.

U_1: If there exist α and β in $N(y)$ such that $y \in \alpha$ and $z \in \beta$, then there exists $\gamma \in N(x)$ such that $z \in \gamma$;

U_2: If there exist α and β in $N(x)$such that $y \in \alpha$ such that $z \in \beta$, then there exists $\gamma \in N(y)$ such that $z \in \gamma$.

As for absoluteness, rules A_1 and A_2 encode the information that for any $x \in W$, given $a \in N(x)$ and $y \in a$, if $\beta \in N(x)$ then $\beta \in N(y)$ (rule A_1), and if $\beta \in N(y)$, then $\beta \in N(x)$ (rule A_2). Thus, $N(x) = N(y)$.

The sequent calculi $\mathbf{G3P.CL}^*$ can be extended modularly to cover Lewis' logics (refer to the end of Section 2). To obtain a calculus for $\mathbb{V}$, it suffices to add to **G3P.CL** a rule corresponding to the semantic condition of nesting:

$$\frac{a \subseteq b, a \in N(x), b \in N(x), \Gamma \Rightarrow \Delta \quad b \subseteq a, a \in N(x), b \in N(x), \Gamma \Rightarrow \Delta}{a \in N(x), b \in N(x), \Gamma \Rightarrow \Delta}\ \mathsf{Nes}$$

The rule can be added to calculi for extensions of $\mathbb{PCL}$ to obtain calculi for the corresponding logics extending $\mathbb{V}$[12].

It might happen that some instances of rules of **G3P.CL*** present a duplication of the atomic formula in the conclusion: for example, an instance of U_1 with $a = b$ displays two formulas $a \in N(x)$ in the conclusion. Since we want contraction to be height-preserving admissible, we deal with these cases by adding to the sequent calculus a new rule, in which the duplicated formulas are contracted into one. Such an operation is called applying a *closure condition* to the rules (cf. [19]). Thus, rule ${\mathsf{U}_1}^*$ is the rule obtained applying the closure condition to U_1 in case $a = b$ and $x = y$; rules ${\mathsf{U}_1}^{**}$ and ${\mathsf{U}_2}^{**}$ are obtained from U_1 and U_2, in case $a = b$ and $y = z$ respectively; and finally, ${\mathsf{A}_1}^*$ is obtained from A_1 in the case $a = b$. Rules ${\mathsf{U}_1}^*$, ${\mathsf{U}_1}^{**}$, ${\mathsf{U}_2}^{**}$ and ${\mathsf{A}_1}^*$ are the only rules we need to define, as other rules defined by closure condition either collapse or are subsumed by other rules of the calculus. For instance, the rule obtained by closure condition to U_2, in case $a = b$ and $x = y$, would be the following:

$$\frac{z \in c, c \in N(x), a \in N(x), x \in a, \Gamma \Rightarrow \Delta}{a \in N(x), x \in a, \Gamma \Rightarrow \Delta}\ {\mathsf{U}_2}^*$$

and this is the same instance we obtain by applying the closure condition to ${\mathsf{U}_1}^*$. However, the rules defined by closure condition are not needed to prove completeness of the calculi; for this reason, we have not included them in the following sections (e.g. in the termination proof).

To prove soundness of the rules with respect to the corresponding system of logics, we need to interpret relational atoms and labelled formulas in neighbourhood models. The notion of realization interprets the labels in neighbourhood frames, thus connecting the syntactic elements of the calculus with the semantic elements of the model.

Definition 4.3. Let S be a sequent. For **K** one of **CL**, **N**, **T**, **W**, **C**, **U**, **A**, **NU**, **TU**, **WU**, **CU**, **NA**, **TA**, **WA**, **CA**, we denote by $\vdash_{\mathsf{G3P.K}} S$ derivability of S in **G3P.K**. Thus, $\vdash_{\mathsf{G3P.CL}} S$ denotes derivability of S in **G3P.CL**, $\vdash_{\mathsf{G3P.N}}$ denotes derivability of S in **G3P.N**, and so on. Since we will need to prove properties of the whole family of labelled calculi, we use **G3P.CL*** to denote any proof system (either **G3P.CL** or one of its extensions) and $\vdash_{\mathsf{G3P.CL^*}} S$ to denote derivability of S in any of such proof systems.

[12] Refer to [14] for a simpler labelled calculus for $\mathbb{V}$, which makes use of the connective of *comparative plausibility* instead of the conditional operator.

Definition 4.4. Let $\mathcal{M}_{\mathcal{K}} = \langle W, N, [\![\ ]\!]\rangle$ be a neighbourhood model for $\mathbb{PCL}$ or one of its extensions, $\mathcal{S}$ a set of world labels and $\mathcal{N}$ a set of neighbourhood labels. An *$\mathcal{SN}$-realization* over $\mathcal{M}_{\mathcal{K}}$ consists of a pair of functions (ρ, σ) such that:

- $\rho : \mathcal{S} \to W$ is the function assigning to each $x \in \mathcal{S}$ an element $\rho(x) \in W$;
- $\sigma : \mathcal{N} \to \mathcal{P}(W)$ is the function assigning to each $a \in \mathcal{N}$ a neighbourhood $\sigma(a) \in N(w)$, for $w \in W$.

We define the notion of *satisfiability of a formula $\mathcal{F}$ at a model $\mathcal{M}_{\mathcal{K}}$ and under a $\mathcal{SN}$-realization* by cases on the form of $\mathcal{F}$:

- $\mathcal{M}_{\mathcal{K}} \vDash_{\rho,\sigma} a \in N(x)$ if $\sigma(a) \in N(\rho(x))$;
- $\mathcal{M}_{\mathcal{K}} \vDash_{\rho,\sigma} a \subseteq b$ if $\sigma(a) \subseteq \sigma(b)$;
- $\mathcal{M}_{\mathcal{K}} \vDash_{\rho,\sigma} y \in \{x\}$ if $\rho(y) \in \sigma(\{x\})$;
- $\mathcal{M}_{\mathcal{K}} \vDash_{\rho,\sigma} x : p$ if $\rho(x) \Vdash p$ [13];
- $\mathcal{M}_{\mathcal{K}} \vDash_{\rho,\sigma} a \Vdash^{\forall} A$ if $\sigma(a) \Vdash^{\forall} A$;
- $\mathcal{M}_{\mathcal{K}} \vDash_{\rho,\sigma} a \Vdash^{\exists} A$ if $\sigma(a) \Vdash^{\exists} A$;
- $\mathcal{M}_{\mathcal{K}} \vDash_{\rho,\sigma} x \Vdash_a A|B$ if $\sigma(a) \in N(\rho(x))$ and for some $\beta \subseteq \sigma(a)$ it holds that $\beta \Vdash^{\exists} A$ and $\beta \Vdash^{\forall} A \to B$;
- $\mathcal{M}_{\mathcal{K}} \vDash_{\rho,\sigma} x : A > B$ if for all $\sigma(a) \in N(\rho(x))$, if $\mathcal{M}_{\mathcal{K}} \vDash_{\rho,\sigma} a \Vdash^{\exists} A$ then $\mathcal{M}_{\mathcal{K}} \vDash_{\rho,\sigma} x \Vdash_a A|B$.

Given a sequent $\Gamma \Rightarrow \Delta$, let $\mathcal{S}$, $\mathcal{N}$ denote, respectively, the sets of world and neighbourhood labels occurring in $\Gamma \cup \Delta$. Let (ρ, σ) be a $\mathcal{SN}$-realization; we say that $\Gamma \Rightarrow \Delta$ *holds at a model $\mathcal{M}_{\mathcal{K}}$ under* (ρ, σ) if the following condition holds: if, for all formulas $C \in \Gamma$, it holds that $\mathcal{M}_{\mathcal{K}} \vDash_{\rho,\sigma} C$, then for some formula $D \in \Delta$ it holds that $\mathcal{M}_{\mathcal{K}} \vDash_{\rho,\sigma} D$. We say that a sequent $\Gamma \Rightarrow \Delta$ is *valid in $\mathcal{M}_{\mathcal{K}}$* if it holds under all $\mathcal{SN}$-realizations (ρ, σ). Finally, a sequent is *valid* in a class of neighbourhood models $\mathcal{K}$ if it is valid in all models $\mathcal{M}_{\mathcal{K}} \in \mathcal{K}$.

Theorem 4.5 (Soundness)**.** *Let S be a sequent and* **K** *one of* **CL**, **N**, **T**, **W**, **C**, **U**, **A**, **NU**, **TU**, **WU**, **CU**, **NA**, **TA**, **WA**, **CA**. *It holds that if $\vdash_{\mathsf{G3P.K}} S$, then S is valid in the $\mathcal{M}_{\mathcal{K}}$ class of neighbourhood models.*

Proof. The proof is by induction on the height of the derivation. We show that the rules preserve validity, employing the notion of *validity of a sequent* defined above. We prove only the cases of $\mathsf{L}>$ and $\mathsf{R}>$.

[$\mathsf{L}>$] Assume that the two premisses of the rule are valid. We have to prove that the conclusion is valid, i.e., that for all the neighbourhood models $\mathcal{M}_{\mathcal{C}l}$ and under all $\mathcal{SN}$-realizations (ρ, σ), it holds that

1. if for all formulas $C \in \Gamma$, $\mathcal{M}_{\mathcal{C}l} \vDash_{\rho,\sigma} C$, and $\mathcal{M}_{\mathcal{C}l} \vDash_{\rho,\sigma} a \in N(x)$, and $\mathcal{M}_{\mathcal{C}l} \vDash_{\rho,\sigma} x : A > B$,
2. then $\mathcal{M}_{\mathcal{C}l} \vDash_{\rho,\sigma} D$, for some formula $D \in \Delta$.

[13]This definition is extended in the standard way to formulas obtained by the classical propositional connectives.

Fix one model $\mathcal{M}_{\mathcal{C}l}$ and an $\mathcal{SN}$-realization (ρ, σ) and assume that 1. holds at $\mathcal{M}_{\mathcal{C}l}$ under (ρ, σ), we have to show that also also 2. holds at $\mathcal{M}_{\mathcal{C}l}$ under (ρ, σ). From the validity of the left premiss of $\mathsf{L}>$ (whence in $\mathcal{M}_{\mathcal{C}l}$) and 1. we have that

3. either $\mathcal{M}_{\mathcal{C}l} \vDash_{\rho,\sigma} D$ for some formula $D \in \Delta$,
4. or $\mathcal{M}_{\mathcal{C}l} \vDash_{\rho,\sigma} a \Vdash^{\exists} A$.

If case 3 holds our statement is proved. Suppose 4 holds. Then, since $\mathcal{M}_{\mathcal{C}l} \vDash_{\rho,\sigma} x : A > B$, we conclude that there exists a $\beta \in \sigma(a)$ such that $\beta \Vdash^{\exists} A$ and $\beta \Vdash^{\forall} A \to B$. Therefore, $\mathcal{M}_{\mathcal{C}l} \vDash_{\rho,\sigma} x \Vdash_a A|B$ and the antecedent of the right premiss holds at $\mathcal{M}_{\mathcal{C}l}$ under (ρ, σ). Since the right premiss is valid, we have that $\mathcal{M}_{\mathcal{C}l} \vDash_{\rho,\sigma} D$ for some formula $D \in \Delta$, thus proving 2.

[$\mathsf{R}>$] Assume the premiss of the rule is valid. We have to show that its conclusion is valid, i.e., that for all the neighbourhood models $\mathcal{M}_{\mathcal{C}l}$ and under all $\mathcal{SN}$-realizations (ρ, σ), it holds that

1. if for all formulas $C \in \Gamma$, $\mathcal{M}_{\mathcal{C}l} \vDash_{\rho,\sigma} C$,
2. then either $\mathcal{M}_{\mathcal{C}l} \vDash_{\rho,\sigma} D$, for some formula $D \in \Delta$,
3. or $\mathcal{M}_{\mathcal{C}l} \vDash_{\rho,\sigma} A > B$.

Fix one model $\mathcal{M}_{\mathcal{C}l}$ and an $\mathcal{SN}$-realization (ρ, σ) and assume that 1 holds at $\mathcal{M}_{\mathcal{C}l}$ under (ρ, σ). If $\mathcal{M} \vDash_{\rho,\sigma} x : A > B$ we are done. Suppose $\mathcal{M}_{\mathcal{C}l} \nvDash_{\rho,\sigma} x : A > B$. Then there is some $\alpha \in N(\rho(x))$ such that $\alpha \Vdash^{\exists} A$ and for all $\beta \subseteq \alpha$, $\beta \nVdash^{\exists} A$ or $\beta \nVdash^{\forall} A \to B$. Since label a doesn't occur in the conclusion of the rule, we can assume without loss of generality that $\sigma(a) = \alpha$. This means that $\mathcal{M} \vDash_{\rho,\sigma} a \Vdash^{\exists} A$ and $\mathcal{M}_{\mathcal{C}l} \nvDash_{\rho,\sigma} x \Vdash_a A|B$. Therefore, by validity of the premiss and hypothesis 1, we conclude that there is a $D \in \Delta$ such that $\mathcal{M}_{\mathcal{C}l} \vDash_{\rho,\sigma} D$, proving 2. □

5 Structural properties and syntactic completeness

In this section we prove the main structural properties of calculi **G3P.CL***, most notably cut-admissibility, (Theorem 5.8) which allows us to give a syntactic proof of completeness for our calculi (Theorem 5.9). We start with some preliminary definitions and lemmas.

Definition 5.1. By *height* of a derivation we mean the number of nodes occurring in the longest derivation branch, minus one. We write $\vdash^n_{\mathsf{G3P.CL}^*} \Gamma \Rightarrow \Delta$ meaning that there is a derivation of $\Gamma \Rightarrow \Delta$ in **G3P.CL*** of height bounded by n.

Definition 5.2. Let r be an inference rule in a proof system S.

$$\frac{P_1, \ldots, P_k}{C}\ \mathsf{r}$$

We define the following notions:

- r is *admissible in* S: if $\vdash_{\mathcal{S}} P_1, \dots, \vdash_{\mathcal{S}} P_k$, then $\vdash_{\mathcal{S}} C$.
- r is *height-preserving admissible in* S: if $\vdash^n_{\mathcal{S}} P_1, \dots, \vdash^n_{\mathcal{S}} P_k$, then $\vdash^n_{\mathcal{S}} C$.
- r is *height-preserving invertible in* S: if $\vdash^n_{\mathcal{S}} C$, then $\vdash^n_{\mathcal{S}} P_1, \dots, \vdash^n_{\mathcal{S}} P_k$.

Definition 5.3. The *weight of relational atoms* is 0. As for the other labelled formulas, the label of formulas of the form $x : A$ and $x \Vdash_a A|B$ is x; the label of formulas $a \Vdash^{\forall} A$ and $a \Vdash^{\exists} A$ is a. We denote by $l(\mathcal{F})$ the label of a formula $\mathcal{F}$, and by $p(\mathcal{F})$ the pure part of the formula, i.e., the part of the formula without the label and without the forcing relation. The *weight of a labelled formula* is defined as a lexicographically ordered pair

$$\langle w(p(\mathcal{F})), w(l(\mathcal{F}))\rangle$$

where

- for all world labels x, $w(x) = 0$;
- for all neighbourhood labels a, $w(a) = 1$;
- $w(p) = w(\bot) = 1$;
- $w(A \circ B) = w(A) + w(B) + 1$ for $\circ$ conjunction, disjunction or implication;
- $w(A|B) = w(A) + w(B) + 2$;
- $w(A > B) = w(A) + w(B) + 3$.

Definition 5.4. The *principal formula* of **G3P.CL*** rules is defined as follows. The principal formula of init is $x : p$, and of $\bot_{\mathsf{L}}$ is $x : \bot$. The principal formula of the rules for local forcing, propositional rules and rules for the conditional is the formula occurring in the conclusion which gets analysed by the rule. Rules for inclusion and rules for extensions do not have a principal formula.

The definition of substitution of labels given in [19] can be extended in an obvious way to the relational atoms and labelled formulas of **G3P.CL***. According to this definition we have, for example, $(a \Vdash^{\exists} A)[b/a] \equiv b \Vdash^{\exists} A$, and $(x \Vdash_a B|A)[y/x] \equiv y \Vdash_a B|A$. The calculus is routinely shown to enjoy the property of height preserving substitution both of world and neighbourhood labels. The proof is a straightforward extension of the same proof in [19].

Proposition 5.5.

(i) *If* $\vdash^n_{\mathsf{G3P.CL}^*} \Gamma \Rightarrow \Delta$, *then* $\vdash^n_{\mathsf{G3P.CL}^*} \Gamma[y/x] \Rightarrow \Delta[y/x]$;
(ii) *If* $\vdash^n_{\mathsf{G3P.CL}^*} \Gamma \Rightarrow \Delta$, *then* $\vdash^n_{\mathsf{G3P.CL}^*} \Gamma[b/a] \Rightarrow \Delta[b/a]$.

The following Lemma, adapted from [19], ensures derivability of generalized initial sequent.

Lemma 5.6. *The following sequents are derivable in* **G3P.CL***.

1. $a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A$
2. $a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta, a \Vdash^{\forall} A$
3. $x \Vdash_a A|B, \Gamma \Rightarrow \Delta, x \Vdash_a A|B$

4. $x : A, \Gamma \Rightarrow \Delta, x : A$

Proof. The four cases are proved by induction on the weight of labelled formulas. All cases are straightforward; by means of example, we prove the first case. Sequent $a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A$ is derived as follows:

$$\dfrac{\dfrac{x \in a, x : A, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A, x : A}{x \in a, x : A, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A}\ \mathsf{R} \Vdash^{\exists}}{a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A}\ \mathsf{L} \Vdash^{\exists}$$

Since $w(x : A) < w(a \Vdash^{\exists} A)$, the upper sequent is derivable by inductive hypothesis, and thus $a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A$ is derivable. □

To prove admissibility of the cut rule, we need to prove the following lemma.

Lemma 5.7. *Let $\mathcal{F}$ be a relational atom or a labelled formula. The following are the rules of* weakening *and* contraction.

$$\dfrac{\Gamma \Rightarrow \Delta}{\mathcal{F}, \Gamma \Rightarrow \Delta}\ \mathsf{wk_L} \quad \dfrac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \mathcal{F}}\ \mathsf{wk_R} \quad \dfrac{\mathcal{F}, \mathcal{F}, \Gamma \Rightarrow \Delta}{\mathcal{F}, \Gamma \Rightarrow \Delta}\ \mathsf{ctr_L} \quad \dfrac{\Gamma \Rightarrow \Delta, \mathcal{F}, \mathcal{F}}{\Gamma \Rightarrow \Delta, \mathcal{F}}\ \mathsf{ctr_R}$$

It holds that:

1. *The rules of weakening are height-preserving admissible in* $\mathbf{G3P.CL}^*$*;*
2. *All the rules of* $\mathbf{G3P.CL}^*$ *are height-preserving invertible;*
3. *The rules of contraction are height-preserving admissible in* $\mathbf{G3P.CL}^*$*.*

Proof. All three statements are proved by induction on the height of the derivation. The proof of 1 amounts to show that if $\vdash^n_{\mathsf{G3P.CL}^*} \Gamma \Rightarrow \Delta$, then $\vdash^n_{\mathsf{G3P.CL}^*} \mathcal{F}, \Gamma \Rightarrow \Delta$ and $\vdash^n_{\mathsf{G3P.CL}^*} \Gamma \Rightarrow \Delta, \mathcal{F}$. This follows straightforwardly using the inductive hypothesis.

To prove 2, we assume that the conclusion of each rule r of $\mathbf{G3P.CL}^*$ is derivable with some fixed derivation height, and show how to derive the premisses of the rule with at most the same derivation height. Invertibility of all structural rules is ensured by weakening. By means of example, we prove two other cases. If $\mathsf{r} = \mathsf{R} \Vdash^{\exists}$, assume $\vdash^n_{\mathsf{G3P.CL}^*} x \in a, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A$. The premiss of the rule is immediately derivable by weakening. If $\mathsf{R} = \mathsf{L} \Vdash^{\exists}$, assume $\vdash^n_{\mathsf{G3P.CL}^*} a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta$. In case $n = 0$, the sequent is an initial sequent; then, also $\vdash^n_{\mathsf{G3P.CL}^*} x \in a, x : A, \Gamma \Rightarrow \Delta$ is an initial sequent. If $n > 0$, the sequent has been derived by some rule. If the rule is $\mathsf{L} \Vdash^{\exists}$, then $\vdash^{n-1}_{\mathsf{G3P.CL}^*} x \in a, x : A, \Gamma \Rightarrow \Delta$ and we are done. If the sequent has been derived by some other rule, apply the inductive hypothesis to the premiss and the rule again to obtain $\vdash^n_{\mathsf{G3P.CL}^*} x \in a, x : A, \Gamma \Rightarrow \Delta$.

To prove 3, suppose $\vdash^n_{\mathsf{G3P.CL}^*} \mathcal{F}, \mathcal{F}, \Gamma \Rightarrow \Delta$. We have to show that $\vdash^n \mathcal{F}, \Gamma \Rightarrow \Delta$. If $n = 0$, the sequent $\mathcal{F}, \mathcal{F}, \Gamma \Rightarrow \Delta$ is an initial sequent; thus, also $\mathcal{F}, \Gamma \Rightarrow \Delta$ is an initial sequent. If $n > 0$, we look at the last rule r applied in the derivation. If $\mathcal{F}$ is not principal in the rule, it suffices to apply the inductive hypothesis to the premiss of r and then r. If $\mathcal{F}$ is the principal formula of r we distinguish two subcases.

3.a) If the principal formula of r appears in the premiss(es) of r ($\mathsf{L} \Vdash^{\forall}$, $\mathsf{R} \Vdash^{\exists}$, $\mathsf{L} >$, $\mathsf{R}|$, and all the structural rules). We apply the inductive hypothesis to the premiss(es), followed by r.

3.b) If the principal formula of r does not appear in the premiss(es) of r (rules for $\wedge$, $\vee$, $\rightarrow$; $\mathsf{R} \Vdash^{\forall}$, $\mathsf{L} \Vdash^{\exists}$, $\mathsf{R} >$, $\mathsf{L}|$), we apply invertibility to the premiss(es) of r, then apply the inductive hypothesis (as many times as needed) followed by r. For instance, suppose $\mathsf{r} = \mathsf{R} >$, $\mathcal{F} = x : A > B$, and

$$\frac{\vdash^{n-1}_{\mathsf{G3P.CL}^*} \; a \in N(x), a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, x \Vdash_a A|B, x : A > B}{\vdash^{n}_{\mathsf{G3P.CL}^*} \; \Gamma \Rightarrow \Delta, x : A > B, x : A > B} \; \mathsf{R} >$$

Applying height-preserving invertibility to the premiss we obtain:

$$\vdash^{n-1}_{\mathsf{G3P.CL}^*} \; a \in N(x), a \in N(x), a \Vdash^{\exists} A, a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, x \Vdash_a A|B, x \Vdash_a A|B.$$

By applying thrice the inductive hypothesis we obtain $\vdash^{n-1}_{\mathsf{G3P.CL}^*} a \in N(x), a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, x \Vdash_a A|B$, and application of $\mathsf{R} >$ to this sequent gives $\vdash^{n}_{\mathsf{G3P.CL}^*} \Gamma \Rightarrow \Delta, x : A > B$.

□

Theorem 5.8 (Cut-admissibility)**.** *The following rule is admissible in* **G3P.CL***.

$$\frac{\Gamma \Rightarrow \Delta, \mathcal{F} \qquad \mathcal{F}, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'} \; \mathsf{cut}$$

Proof. The proof proceeds by primary induction on the weight of the cut formula and secondary induction on the sum of the heights of the derivations of the premisses of cut. We assume that the premisses of cut are derivable in **G3P.CL***, and show how the conclusion of cut can be derived. We distinguish cases according to the last rule applied in the derivation of the two premisses[14]:

a) At least one of the premisses of cut is an initial sequent;
b) $\mathcal{F}$ is not the principal formula in the derivation of at least one premiss;
c) $\mathcal{F}$ is the principal formula of both derivations of the premisses.

Height-preserving admissibility of weakening is needed to prove case *a*), and case *b*) follows quite immediately from the inductive hypothesis. The interested reader is referred to [10, Theorem 3.4.6] for a proof of these cases. We prove only case *c*) for $\mathcal{F} \equiv x \Vdash_a A|B$ and $\mathcal{F} \equiv x : A > B$. The standard proofs for $\mathcal{F} = x : A \wedge B$, $\mathcal{F} = x : A \vee B$ and $\mathcal{F} = x : A \rightarrow B$ and can be found in [22, Theorem 3.2.3]; the proofs for $\mathcal{F} = a \Vdash^{\exists} A$ and $\mathcal{F} = a \Vdash^{\forall} A$ follow a similar strategy as the cases shown here and can be found in [10, Theorem 3.4.6].

[14]Refer to [22] for the general methodology of proving cut-admissibility in labelled systems.

Case c) $\mathcal{F} \equiv x \Vdash_a A|B$

$$\dfrac{\dfrac{\overset{(1)}{} \qquad \overset{(2)}{}}{c \in N(x), c \subseteq a, \Gamma \Rightarrow \Delta, x \Vdash_a A|B}\ \mathsf{R|} \qquad \dfrac{(3)}{x \Vdash_a A|B, \Gamma' \Rightarrow \Delta'}\ \mathsf{L|}}{c \in N(x), c \subseteq a, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \mathsf{cut}$$

Where:

(1) is $c \in N(x), c \subseteq a, \Gamma \Rightarrow \Delta, x \Vdash_a A|B, c \Vdash^{\exists} A$;

(2) is $c \in N(x), c \subseteq a, \Gamma \Rightarrow \Delta, x \Vdash_a A|B, c \Vdash^{\forall} A \to B$

(3) is $d \in N(x), d \subseteq a, d \Vdash^{\exists} A, d \Vdash^{\forall} A \Rightarrow B, \Gamma' \Rightarrow \Delta'$.

First, we apply cut on premisses of $\mathsf{R|}$. These applications of cut have sum of height of the premisses smaller than the two premisses of cut:

$$\mathcal{D}_1 \quad = \quad \dfrac{(1) \qquad x \Vdash_a A|B, \Gamma' \Rightarrow \Delta'}{c \in N(x), c \subseteq a, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', c \Vdash^{\exists} A}\ \mathsf{cut}$$

$$\mathcal{D}_2 \quad = \quad \dfrac{(2) \qquad x \Vdash_a A|B, \Gamma' \Rightarrow \Delta'}{c \in N(x), c \subseteq a, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', c \Vdash^{\forall} A \to B}\ \mathsf{cut}$$

Then we apply two further cuts, on formulas of a smaller weight than $\mathcal{F}$. In the following derivations, ctr^* denotes several occurrences of $\mathsf{ctr_L}$ and $\mathsf{ctr_R}$ and the superscripts next to labelled formulas or multisets denote the number of occurrences of that formula or multiset in the sequent.

$$\dfrac{\dfrac{\mathcal{D}_2 \qquad \dfrac{\mathcal{D}_1 \qquad \overset{(3)[d/c]}{c \in N(x), c \subseteq a, c \Vdash^{\exists} A, c \Vdash^{\forall} A \Rightarrow B, \Gamma' \Rightarrow \Delta'}}{c \in N(x)^2, c \subseteq a^2, \Gamma, \Gamma'^2 \Rightarrow \Delta, \Delta'^2, c \Vdash^{\forall} A \to B}\ \mathsf{cut}}{c \in N(x)^3, c \subseteq a^3, \Gamma^2, \Gamma'^3 \Rightarrow \Delta^2, \Delta'^3}\ \mathsf{cut}}{c \in N(x), c \subseteq a, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \mathsf{ctr}^*$$

Case c) $\mathcal{F} \equiv x : A > B$

$$\dfrac{\dfrac{\overset{(1)}{b \in N(x), b \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, x \Vdash_a A|B}}{\Gamma \Rightarrow \Delta, x : A > B}\ \mathsf{R>} \qquad \dfrac{\overset{(2)}{.. \Rightarrow \Delta', a \Vdash^{\exists} A} \quad \overset{(3)}{x \Vdash_a A|B, a \in N(x), x : A > B, \Gamma' \Rightarrow \Delta'}}{a \in N(x), x : A > B, \Gamma' \Rightarrow \Delta'}\ \mathsf{L>}}{a \in N(x), \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \mathsf{cut}$$

We first apply cut on the premisses of $\mathsf{L>}$. Both applications have a smaller sum of height of the premisses with respect to the premisses of cut:

$$\mathcal{D}_1 = \quad \dfrac{\Gamma \Rightarrow \Delta, x : A > B \quad \overset{(2)}{a \in N(x), x : A > B, \Gamma' \Rightarrow \Delta', a \Vdash^{\exists} A}}{a \in N(x), \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \mathsf{cut}$$

$$\mathcal{D}_2 = \quad \dfrac{\Gamma \Rightarrow \Delta, x : A > B \quad \overset{(3)}{x \Vdash_a A|B, a \in N(x), x : A > B, \Gamma' \Rightarrow \Delta'}}{a \in N(x), \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \mathsf{cut}$$

We combine $\mathcal{D}_1$ and $\mathcal{D}_2$ by means of two occurrences of cut as follows:

$$\dfrac{\dfrac{\mathcal{D}_1 \qquad \overset{(1)[b/a]}{b \in N(x), b \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, x \Vdash_a A|B}}{a \in N(x)^2, \Gamma^2, \Gamma' \Rightarrow \Delta^2, \Delta', x \Vdash_a A|B}\ \mathsf{cut} \qquad \mathcal{D}_2}{\dfrac{a \in N(x)^3, \Gamma^3, \Gamma'^2 \Rightarrow \Delta^3, \Delta'^2}{a \in N(x), \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \mathsf{ctr}^*}\ \mathsf{cut}$$

The two occurrences of cut are on formulas of lesser weight than $\mathcal{F}$, and thus justified by the inductive hypothesis. $\square$

The axioms of each system of logic can be derived in the respective calculus. By admissibility of cut, the inference rules can be shown to be admissible, therefore obtaining a syntactic proof of completeness of the calculi. Details are given in the Appendix.

Theorem 5.9 (Completeness via cut admissibility)**.** *Let $F \in \mathcal{L}$, and* **K** *one of* **CL***,* **N***,* **T***,* **W***,* **C***,* **U***,* **A***,* **NU***,* **TU***,* **WU***,* **CU***,* **NA***,* **TA***,* **WA***,* **CA***. It holds that if $\vdash_{\mathsf{PK}} F$, then $\vdash_{\mathsf{G3P.K}} \Rightarrow x : F$.*

We conclude the section by proving admissibility of rules repl_1 and repl_2 in their generalized form. This lemma will be used in Section 7, to prove completeness of the calculi featuring centering with respect to neighbourhood models.

Lemma 5.10. *Rules repl_1 and repl_2 generalized to all formulas of the language are admissible in* **G3P.CL****.*

Proof. Admissibility of the two rules is proven simultaneously, by induction on the weight of formulas. We only show the proof admissibility for repl_1 (the other rule is symmetric). Since contraction and cut are admissible in **G3P.CL***, it is sufficient to show that sequent $y \in \{x\}, A(x) \Rightarrow A(y)$ is derivable. From this sequent and the premiss of repl_1, the conclusion of repl_1 can be derived applying cut and contraction. We proceed by induction on the weight of formula $A(x)$; there are several cases to consider.
1. $A(x) \equiv x : \mathcal{F}$, $A(y) \equiv y : \mathcal{F}$, where $\mathcal{F}$ is a propositional formula. We consider the case $A(x) \equiv x : B \to C$, $A(y) \equiv y : B \to C$.

$$\dfrac{\dfrac{\dfrac{y \in \{x\}, x : B, y : B \Rightarrow y : C, x : B}{y \in \{x\}, y : B \Rightarrow y : C, x : B}\ \mathsf{repl}_2 \qquad \dfrac{y \in \{x\}, y : B, x : C, y : C \Rightarrow y : C, x : B}{y \in \{x\}, y : B, x : C \Rightarrow y : C}\ \mathsf{repl}_1}{y \in \{x\}, x : B \to C, y : B \Rightarrow y : C}\ \mathsf{L}\to}{y \in \{x\}, x : B \to C \Rightarrow y : B \to C}\ \mathsf{R}\to$$

In this case we need repl_2, applied to formulas of smaller weight, and the two premisses are derivable by Lemma 5.6.

2. $A(x) \equiv x \Vdash_a B|C$, $A(y) \equiv y \Vdash_a B|C$.

$$\dfrac{\dfrac{\dfrac{(1) \qquad (2)}{c \in N(y), c \in N(x), c \subseteq a, c \Vdash^{\exists} B, c \Vdash^{\forall} B \to C, y \in \{x\} \Rightarrow y \Vdash_a B|C}\ \mathsf{R|}}{c \in N(x), c \subseteq a, c \Vdash^{\exists} B, c \Vdash^{\forall} B \to C, y \in \{x\} \Rightarrow y \Vdash_a B|C}\ \mathsf{repl_1}}{y \in \{x\}, x \Vdash_a B|C \Rightarrow y \Vdash_a B|C}\ \mathsf{L|}$$

Where (1) is sequent $c \in N(y), c \in N(x), c \subseteq a, c \Vdash^{\exists} B, c \Vdash^{\forall} B \to C, y \in \{x\} \Rightarrow y \Vdash_a B|C, c \Vdash^{\exists} A$, and (2) is sequent $c \in N(y), c \in N(x), c \subseteq a, c \Vdash^{\exists} B, c \Vdash^{\forall} B \to C, y \in \{x\} \Rightarrow y \Vdash_a B|C, c \Vdash^{\forall} B \to C$. Rule $\mathsf{repl_1}$ is applied to the atomic formula $c \in N(x)$, which has smaller weight than $A(x)$. The lower premiss is derivable by Lemma 5.6, the upper one by steps of $\mathsf{L} \Vdash^{\exists}$, $\mathsf{L} \Vdash^{\forall}$, $\mathsf{L} \subseteq$, and Lemma 5.6.

3. $A(x) \equiv x : B > C$, $A(y) \equiv y : B > C$.

$$\dfrac{\dfrac{\dfrac{x \Vdash_a B|C, a \in N(x), a \in N(y), a \Vdash^{\exists} B, y \in \{x\}, x : B > C \Rightarrow y \Vdash_a B|C}{a \in N(x), a \in N(y), a \Vdash^{\exists} B, y \in \{x\}, x : B > C \Rightarrow y \Vdash_a B|C}\ \mathsf{L>}}{a \in N(y), a \Vdash^{\exists} B, y \in \{x\}, x : B > C \Rightarrow y \Vdash_a B|C}\ \mathsf{repl_2}}{y \in \{x\}, x : B > C \Rightarrow y : B > C}\ \mathsf{R>}$$

Rule $\mathsf{repl_2}$ is applied to formula $a \in N(y)$, of smaller weight. The leftmost premiss is the sequent $a \in N(x), a \in N(y), a \Vdash^{\exists} B, y \in \{x\}, x : B > C \Rightarrow y \Vdash_a B|C, a \Vdash^{\exists} A$, derivable by Case 1. □

6 Decision procedure

As they are, the calculi **G3P.CL*** are not terminating. Simple cases of loops are due to the repetition of the principal formula in the premiss of a rule; more complex cases of loop are generated by the interplay of world and neighbourhood labels. Our aim in this section is to provide a termination strategy for the calculi, thus defining a decision procedure for the logic.

Here follow some examples of loops which might occur in root-first proof search.

Example 6.1. *Loop generated by repeated applications of rule* $\mathsf{L} \Vdash^{\forall}$ *to* $a \Vdash^{\forall} C$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{x \in a, x : A, x : C, x : C, a \Vdash^{\forall} C, \Gamma \Rightarrow \Delta}}{x \in a, x : A, x : C, a \Vdash^{\forall} C, \Gamma \Rightarrow \Delta}\ \mathsf{L} \Vdash^{\forall}}{x \in a, x : A, a \Vdash^{\forall} C, \Gamma \Rightarrow \Delta}\ \mathsf{L} \Vdash^{\forall}}{a \Vdash^{\exists} A, a \Vdash^{\forall} C, \Gamma \Rightarrow \Delta}\ \mathsf{L} \Vdash^{\exists}$$

Example 6.2. *Loop generated by repeated applications of* $\mathsf{L}>$ *and* $\mathsf{L|}$*, with one conditional formula in the antecedent (only the left premiss of* $\mathsf{L}>$ *is shown).*

$$
\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ c \in N(x), c \subseteq b, b \in N(x), b \subseteq a, a \in N(x), c \Vdash^{\exists} A, c \Vdash^{\forall} A \to B, \dots, x : A > B \Rightarrow \Delta\end{array}}{x \Vdash_b A|B, b \in N(x), b \subseteq a, a \in N(x), b \Vdash^{\exists} A, b \Vdash^{\forall} A \to B, x : A > B \Rightarrow \Delta}\ \mathsf{L}|}{b \in N(x), b \subseteq a, a \in N(x), b \Vdash^{\exists} A, b \Vdash^{\forall} A \to B, x : A > B \Rightarrow \Delta}\ \mathsf{L}>}{x \Vdash_a A|B, a \in N(x), x : A > B \Rightarrow \Delta}\ \mathsf{L}|}{a \in N(x), x : A > B \Rightarrow \Delta}\ \mathsf{L}>
$$

Example 6.3. *Loop generated by repeated applications of rules* $\mathsf{L}>$ *and* $\mathsf{L}|$*, with two conditional formulas in the antecedent. Let* $\Omega = x : A > B, x : C > D$*. We write only the leftmost premiss of* $\mathsf{L}>$*; next to* $\mathsf{L}>$ *is written the number of applications of the rule.*

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ x \Vdash_d A|B, x \Vdash_d C|D, x \Vdash_e A|B, x \Vdash_e C|D, x \Vdash_f A|B, x \Vdash_f C|D, x \Vdash_g A|B, x \Vdash_g C|D, \dots, \Omega \Rightarrow \Delta\end{array}}{g \in N(x), g \subseteq c, g \Vdash^{\exists} C, g \Vdash^{\forall} C \to D, \dots, \Omega \Rightarrow \Delta}\ \mathsf{L}>\ \textit{(4)}}{f \in N(x), f \subseteq c, f \Vdash^{\exists} A, f \Vdash^{\forall} A \to B, x \Vdash_c C|D, \dots, \Omega \Rightarrow \Delta}\ \mathsf{L}|}{e \in N(x), e \subseteq b, e \Vdash^{\exists} C, e \Vdash^{\forall} C \to D, x \Vdash_c A|B, x \Vdash_c C|D, \dots, \Omega \Rightarrow \Delta}\ \mathsf{L}|}{d \in N(x), d \subseteq b, d \Vdash^{\exists} A, d \Vdash^{\forall} A \to B, x \Vdash_b C|D, x \Vdash_c A|B, x \Vdash_c C|D, \dots, \Omega \Rightarrow \Delta}\ \mathsf{L}|}{x \Vdash_b A|B, x \Vdash_b C|D, x \Vdash_c A|B, x \Vdash_c C|D, \dots, \Omega \Rightarrow \Delta}\ \mathsf{L}|}{c \in N(x), c \subseteq a, c \Vdash^{\exists} C, c \Vdash^{\forall} C \to D, \dots, \Omega \Rightarrow \Delta}\ \mathsf{L}>\ \textit{(4)}}{b \subseteq a, b \in N(x), b \Vdash^{\exists} A, b \Vdash^{\forall} A \to B, x \Vdash_a C|D, \Omega \Rightarrow \Delta}\ \mathsf{L}|}{x \Vdash_a A|B, x \Vdash_a C|D, \Omega \Rightarrow \Delta}\ \mathsf{L}|}{a \in N(x), \Omega \Rightarrow \Delta}\ \mathsf{L}>\ \textit{(2)}
$$

We start by proving termination for **G3P.CL**, and then extend the proof strategy to sequent calculi for the extensions of $\mathbb{PCL}$.

Remark 6.4. All conditional logics treated in this work are decidable, and their complexity is studied in [5]. For systems *without* uniformity and absoluteness, the decision procedure is PSPACE-complete. For logics with uniformity, the decision problem is EXPTIME-complete. Finally, for systems with absoluteness, the decision problem is NP-complete.

6.1 Decidability for G3P.CL

In this section we define a proof search strategy which blocks rules applications leading to non-terminating branches. We first want to prevent applications of a rule r to a sequent that already contains the formulas introduced by r. This is done by defining saturation conditions for each rule.

Definition 6.5. Let $\mathcal{D}$ be a derivation in **G3P.CL**, and $\mathcal{B} = S_0, S_1, \dots$ a derivation branch, with S_i sequent $\Gamma_i \Rightarrow \Delta_i$, for $i = 1, 2, \dots$ and S_0 sequent $\Rightarrow x : A_0$. Let $\downarrow \Gamma_k$, resp. $\downarrow \Delta_k$, denote the union of the antecedents, resp. succedents, occurring in the branch from S_0 up to S_k.

We say that a sequent $\Gamma \Rightarrow \Delta$ *satisfies the saturation condition w.r.t. a rule* r if, whenever $\Gamma \Rightarrow \Delta$ contains the principal formulas in the conclusion of r, then it also contains the formulas introduced by *one* of the premisses of r. The saturation conditions are listed in Figure 6.

L$\wedge$	If $x : A \wedge B$ is in $\downarrow\Gamma$, then $x : A$ and $x : B$ are in $\downarrow\Gamma$
R$\wedge$	If $x : A \wedge B$ is in $\downarrow\Delta$, then $x : A$ or $x : B$ is in $\downarrow\Delta$
L$\vee$	If $x : A \vee B$ is in $\downarrow\Gamma$, then $x : A$ or $x : B$ is in $\downarrow\Gamma$
R$\vee$	If $x : A \vee B$ is in $\downarrow\Delta$, then $x : A$ and $x : B$ are in $\downarrow\Delta$
L $\to$	If $x : A \to B$ is in $\downarrow\Gamma$, then $x : B$ is in $\downarrow\Gamma$ or $x : A$ is in $\downarrow\Delta$
R $\to$	If $x : A \to B$ is in Δ, then $x : A$ is in $\downarrow\Gamma$ and $x : B$ is in $\downarrow\Delta$
ref	If a is in $\Gamma \cup \Delta$, Δ then $a \subseteq a$ is in Γ
tr	If $a \subseteq b$ and $b \subseteq c$ are in Γ, then $a \subseteq c$ is in Γ
L $\subseteq$	If $x \in a$ and $a \subseteq b$ are in Γ, then $x \in b$ is in Γ
L $\Vdash^{\forall}$	If $x \in a$ and $a \Vdash^{\forall} A$ are in Γ, then $x : A$ is in $\downarrow\Gamma$
R $\Vdash^{\forall}$	If $a \Vdash^{\forall} A$ is in $\downarrow\Delta$ then, for some x, $x \in a$ is in Γ and $x : A$ in $\downarrow\Delta$
L $\Vdash^{\exists}$	If $a \Vdash^{\exists} A$ is in $\downarrow\Gamma$ then, for some x, $x \in a$ is in Γ and $x : A$ is in $\downarrow\Gamma$
R $\Vdash^{\exists}$	If $x \in a$ is in Γ and $a \Vdash^{\exists} A$ is in Δ, then $x : A$ is in $\downarrow\Delta$
R $>$	If $x : A > B$ is in $\downarrow\Delta$ then, for some a, $a \in N(x)$ is in Γ, $a \Vdash^{\exists} A$ is in $\downarrow\Gamma$ and $x \Vdash_a A\|B$ is in Δ
L $>$	If $a \in N(x)$ and $x : A > B$ are in Γ, then $a \Vdash^{\exists} A$ is in $\downarrow\Delta$ or $x \Vdash_a B\|A$ is in $\downarrow\Gamma$
R$\|$	If $c \in N(x)$ and $c \subseteq a$ are in Γ and $x \Vdash_a B\|A$ is in Δ, then $c \Vdash^{\exists} A$ is in Δ or $c \Vdash^{\forall} A \to B$ is in $\downarrow\Delta$
L$\|$	If $x \Vdash_a B\|A$ is in $\downarrow\Gamma$ then, for some c, $c \in N(x)$ and $c \subseteq a$ are in Γ, $c \Vdash^{\exists} A$ is in $\downarrow\Gamma$ and $c \Vdash^{\forall} A \to B$ is in Γ
L $>^{\star}$	If $a \in N(x)$ and $x : A > B$ occur in Γ, then $a \Vdash^{\exists} A$ is in $\downarrow\Delta$ or $a \Vdash^{\exists} A$ and $x \Vdash_a B\|A$ are in $\downarrow\Gamma$
Mon$\forall$	If $b \subseteq a$ and $a \Vdash^{\forall} A$ are in Γ, then $b \Vdash^{\forall} A$ is in Γ

Figure 6: Saturation conditions associated to **G3P.CL** rules

We say that $\Gamma \Rightarrow \Delta$ is *saturated* if there is no formula $x : p$ occurring in $\Gamma \cap \Delta$, there is no formula $x : \bot$ occurring in Γ, and $\Gamma \Rightarrow \Delta$ satisfies *all* saturation conditions listed in the upper part of Figure 6.

In Example 6.1, the second bottom-up application of L $\Vdash^{\forall}$ is blocked by the saturation condition associated to L $\Vdash^{\forall}$, since formula $x : A$ already occurs in some antecedent of the derivation branch. In order to block the other cases of loop, we need to modify the rules of **G3P.CL**, and define a proof search strategy which governs the application of rules in root-first proof search.

Definition 6.6. We modify the rule L $>$ into rule L $>^{\star}$, and introduce rule Mon$\forall$ in **G3P.CL**.

$$\frac{a \in N(x), x : A > B, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A \quad a \Vdash^{\exists} A, x \Vdash_a A|B, a \in N(x), x : A > B, \Gamma \Rightarrow \Delta}{a \in N(x), x : A > B, \Gamma \Rightarrow \Delta}\ \mathsf{L}>^{\star}$$

$$\frac{b \subseteq a, b \Vdash^{\forall} A, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}{b \subseteq a, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}\ \mathsf{Mon}\forall$$

Lemma 6.7. *In* **G3P.CL** *it holds that:*

1. *Rule* $\mathsf{Mon}\forall$ *is admissible;*

2. *Rules* $\mathsf{L}>$ *and* $\mathsf{L}>^{\star}$ *are equivalent.*

Proof. The proof of 1 is immediate, by induction on the height of the derivation. To prove that $\mathsf{L}>^{\star}$ is admissible if we have $\mathsf{L}>$, apply weakening to the right premiss of $\mathsf{L}>$ and then apply $\mathsf{L}>^{\star}$ to obtain the conclusion of $\mathsf{L}>$. To prove that $\mathsf{L}>$ is admissible if we have $\mathsf{L}>^{\star}$, we need admissibility of cut. Let (1) and (2) denote the left and right premiss of $\mathsf{L}>^{\star}$. The conclusion of $\mathsf{L}>^{\star}$ is derived as follows:

$$\dfrac{(1) \qquad \dfrac{\dfrac{(1)}{x \Vdash_a A|B, a \in N(x), x : A > B, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A}\ \mathsf{Wk_L} \qquad (2)}{x \Vdash_a A|B, a \in N(x), x : A > B, \Gamma \Rightarrow \Delta}\ \mathsf{cut}}{a \in N(x), x : A > B, \Gamma \Rightarrow \Delta}\ \mathsf{L}>$$

□

Definition 6.8. The saturation conditions for $\mathsf{L}>^{\star}$ and $\mathsf{Mon}\forall$ are defined in the lower part of Figure 6. The list of saturation conditions needed for the termination proof is given by the conditions listed in the upper part of Figure 6, in which the condition $\mathsf{L}>$ is replaced by $\mathsf{L}>^{\star}$, and the saturation condition for $\mathsf{Mon}\forall$ is added.

We shall provide a decision procedure for sequent calculus **G3P.CL** modified with rules $\mathsf{Mon}\forall$ and $\mathsf{L}>^{\star}$. We start by defining the proof search strategy.

Definition 6.9. When constructing root-first a derivation tree for a sequent $\Rightarrow x_0 : A$, apply the following *proof search strategy*:

1. Apply rules which introduce a new label (*dynamic* rules) only if rules which do not introduce a new label (*static* rules) are not applicable; as an exception, apply $\mathsf{R}>$ before $\mathsf{L}>^{\star}$.

2. If a sequent satisfies a saturation condition r, do not apply to that sequent the rule r corresponding to the saturation condition.

Observe that if the strategy is applied, world labels in root-first proof search are processed one after the other, according to the order in which they are generated.

Example 6.10. *In Example 6.2, the loop is stopped thanks to the proof search strategy and the saturation condition for* $\mathsf{L}|$*, which blocks the uppermost application of the rule to the formula* $x \Vdash_b A|B$*. The proof strategy requires static rules to be introduced* before *dynamic rules. Thus, the static rule* ref *is applied* before *the uppermost occurrence of the dynamic rule* $\mathsf{L}|$*, introducing in the derivation formula* $b \subseteq b$[15]*. The saturation condition for* $\mathsf{L}|$ *applied to* $x \Vdash_b A|B$ *is met if there is some label* d *such that formulas* $d \subseteq b$*,* $d \in N(x)$*,* $d \Vdash^{\exists} A$ *and*

[15]By a similar argument, also $a \subseteq a$ and a number of other formulas should occur in the derivation before the uppermost application of $\mathsf{L}|$; but they are not relevant here.

$d \Vdash^{\forall} A \to B$ already occur in the antecedent of a sequent occurring lower in the branch. Thus, if we take d to be b itself, the saturation condition is met and the uppermost occurrence of $\mathsf{L}|$ cannot be applied.

To see how the loop in Example 6.3 is stopped, we re-write the derivation according to the proof search strategy, highlighting the formulas to which rule $\mathsf{L}|$ cannot be applied. Observe that here rule $\mathsf{L}>^{\star}$ and $\mathsf{Mon}\forall$ become relevant. The same conventions as in Example 6.3 apply.

$$
\begin{prooftree}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$d \Vdash^{\exists} A, d \Vdash^{\exists} C, e \Vdash^{\exists} A, e \Vdash^{\exists} C, \mathbf{x} \Vdash_{\mathbf{d}} \mathbf{A}|\mathbf{B}, \mathbf{x} \Vdash_{\mathbf{d}} \mathbf{C}|\mathbf{D}, \mathbf{x} \Vdash_{\mathbf{e}} \mathbf{A}|\mathbf{B}, \mathbf{x} \Vdash_{\mathbf{e}} \mathbf{C}|\mathbf{D}, \dots, \Omega \Rightarrow \Delta$}
\RightLabel{$\mathsf{L}>^{\star}$ \textit{(4)}}
\UnaryInfC{$d \subseteq d, e \subseteq e, e \in N(x), e \subseteq c, e \Vdash^{\exists} A, e \Vdash^{\forall} A \to B, e \Vdash^{\forall} C \to D, \dots, \Omega \Rightarrow \Delta$}
\RightLabel{ref \textit{(2)}}
\UnaryInfC{$e \in N(x), e \subseteq c, e \Vdash^{\exists} A, e \Vdash^{\forall} A \to B, e \Vdash^{\forall} C \to D, \dots, \Omega \Rightarrow \Delta$}
\RightLabel{$\mathsf{Mon}\forall$}
\UnaryInfC{$e \in N(x), e \subseteq c, e \Vdash^{\exists} A, e \Vdash^{\forall} A \to B, \dots, \Omega \Rightarrow \Delta$}
\RightLabel{$\mathsf{L}|$}
\UnaryInfC{$d \in N(x), d \subseteq b, d \Vdash^{\exists} C, d \Vdash^{\forall} C \to D, d \Vdash^{\forall} A \to B, x \Vdash_c A|B, \dots, \Omega \Rightarrow \Delta$}
\RightLabel{$\mathsf{Mon}\forall$}
\UnaryInfC{$d \in N(x), d \subseteq b, d \Vdash^{\exists} C, d \Vdash^{\forall} C \to D, x \Vdash_c A|B, \dots, \Omega \Rightarrow \Delta$}
\RightLabel{$\mathsf{L}|$}
\UnaryInfC{$b \Vdash^{\exists} A, b \Vdash^{\exists} C, c \Vdash^{\exists} A, c \Vdash^{\exists} C, \mathbf{x} \Vdash_{\mathbf{b}} \mathbf{A}|\mathbf{B}, x \Vdash_b C|D, x \Vdash_c A|B, \mathbf{x} \Vdash_{\mathbf{c}} \mathbf{C}|\mathbf{D}, \dots, \Omega \Rightarrow \Delta$}
\RightLabel{$\mathsf{L}>^{\star}$ \textit{(4)}}
\UnaryInfC{$b \subseteq b, c \subseteq c, c \in N(x), c \subseteq a, c \Vdash^{\exists} C, c \Vdash^{\forall} C \to D, \dots, \Omega \Rightarrow \Delta$}
\RightLabel{ref \textit{(2)}}
\UnaryInfC{$c \in N(x), c \subseteq a, c \Vdash^{\exists} C, c \Vdash^{\forall} C \to D, \dots, \Omega \Rightarrow \Delta$}
\RightLabel{$\mathsf{L}|$}
\UnaryInfC{$b \subseteq a, b \in N(x), b \Vdash^{\exists} A, b \Vdash^{\forall} A \to B, x \Vdash_a C|D, \Omega \Rightarrow \Delta$}
\RightLabel{$\mathsf{L}|$}
\UnaryInfC{$a \Vdash^{\exists} A, a \Vdash^{\exists} C, x \Vdash_a A|B, x \Vdash_a C|D, \Omega \Rightarrow \Delta$}
\RightLabel{$\mathsf{L}>^{\star}$ \textit{(2)}}
\UnaryInfC{$a \in N(x), \Omega \Rightarrow \Delta$}
\end{prooftree}
$$

Application of $\mathsf{L}|$ to formula $x \Vdash_b A|B$ is blocked by the saturation condition, since $b \subseteq b$, $b \in N(x)$, $b \Vdash^{\exists} A$ and $b \Vdash^{\forall} A \to B$ all occur in the branch. Application of the rule to $x \Vdash_c C|D$ is blocked in a similar way. Application of $\mathsf{L}|$ to $x \Vdash_d A|B$ is blocked, since all the formulas relevant for the saturation condition occur in the branch: $d \subseteq d$ (introduced by ref), $d \in N(x)$, $d \Vdash^{\exists} A$ (introduced by $\mathsf{L}>^{\star}$) and $d \Vdash^{\forall} A \to B$ (introduced by $\mathsf{Mon}\forall$). Application of $\mathsf{L}|$ to the other formulas in the top sequent is blocked, and the loop is stopped.

Before tackling the termination proof, we define an ordering of the world labels according to their generation in the branch. The resulting tree of labels is needed to ensure that the number formulas introduced in root-first proof search is *finite*.

Definition 6.11. Given a sequent $\Gamma_k \Rightarrow \Delta_k$, let a, b be neighbourhood labels and x, y world labels occurring in $\downarrow \Gamma_k \cup \downarrow \Delta_k$. We define:

- $k(x) = min\{t \mid x \text{ occurs in } \Gamma_t\}$;
- $k(a) = min\{t \mid a \text{ occurs in } \Gamma_t\}$;
- $x \prec_g a$, "x generates a" if for some $t \leqslant k$ and $k(a) = t$, $a \in N(x)$ occurs in Γ_t;
- $b \prec_g y$, "b generates y" if for some $t \leqslant k$ and $k(y) = t$, $y \in b$ occurs in Γ_t;
- $x \prec y$ "x is an ancestor of y" if for some a, $x \prec_g a$ and $a \prec_g y$ and $x \neq y$.

Intuitively, the relation $x \prec_g a$ holds between x and a if $a \in N(x)$ is introduced at some stage in the derivation (thus, with an application of $\mathsf{R}>$ or $\mathsf{L}|$); similarly, the relation $b \prec_g y$ holds between b and y if $y \in b$ is introduced in the derivation (thus, applying either $\mathsf{R} \Vdash^{\forall}$ or $\mathsf{L} \Vdash^{\exists}$).

Lemma 6.12. *Given a branch of a derivation as described in Definition 6.5, the following hold:*

(a) The relation $\prec$ is acyclic and forms a tree with the world label x_0 at the root;
(b) All labels occurring in a derivation branch also occur in the associated tree; that is, letting $x \prec^* y$ be the transitive closure of $\prec$, if u occurs in $\downarrow \Gamma_k$, then $x_0 \prec^* u$.

Proof. (a) follows from the definition of relation $\prec$ and from the sequent calculus rules. Observe that the relation $\prec_g$ between world and neighbourhood labels is *unique*: it is defined by taking into account the value $k(a)$ or $k(y)$, which keeps track of the derivation step at which the new label is introduced. At each derivation step, dynamic rules introduce a new label which is generated by at most one world or neighbourhood label. Take a sequent $\Gamma_k \Rightarrow \Delta_k$, and suppose $x \prec_g a$: by definition, there is a $t \leqslant k$ such that $k(a) = t$ and $a \in N(x)$ occurs in Γ_t. Now suppose that $a \in N(y)$ occurs in some Γ_s, with $t < s \leqslant k$. Since $k(a) = t$, relation $y \prec_g a$ does not hold in the tree of labels. A similar reasoning holds for $y \prec_g a$. Thus, except for the label at the root, each label in a derivation branch has *exactly* one parent according to the relation $\prec_g$ and, by definition, also according to $\prec$.

As for (b), it is easily proved by induction on $k(u) \leqslant k$. If $k(u) = 0$, then $u = x_0$ and (b) trivially holds. If $k(u) = t > 0$, u does not occur in Γ_{t-1} and u occurs in Γ_t. This means that there exist a v and a b such that $b \in N(v)$ occurs in Γ_{t-1} and $u \in b$ occurs in Γ_t; thus, $k(v) < k(u)$. By inductive hypothesis, $x_0 \prec^* v$; since $v \prec u$, also $x_0 \prec^* u$ holds. □

Definition 6.13. For this definition, we take into account only labelled formulas of the form $x : A$. The *size* of a formula A, denoted by $|A|$, is the number of symbols occurring in A.

The *conditional degree* of a formula A corresponds to the level of nesting of the conditional operator in A and is defined as follows:

- $d(p) = d(\bot) = 0$ for p atomic;
- $d(C \circ D) = max(d(C), d(D))$ for $\circ \in \{\wedge, \vee, \to\}$;
- $d(C > D) = max(d(C), d(D)) + 1$.

Given a sequent $\Gamma \Rightarrow \Delta$ occurring in a derivation branch $\mathcal{B}$, the conditional degree of a world label x is the highest conditional degree among the formulas it labels:

$$d(x) = max\{d(C) \mid x : C \in \downarrow \Gamma \cup \downarrow \Delta\}.$$

We now prove that the proof search strategy ensures termination.

Theorem 6.14 (Termination)**.** *Root-first proof search for a* **G3P.CL** *derivation of sequent* $\Rightarrow x_0 : A_0$ *built in accordance with the strategy terminates in a finite number of steps, with each leaf of the derivation tree containing either an initial sequent or a saturated sequent.*

Proof. To prove that root-first proof search terminates, we have to show that all the branches of a derivation starting with $\Rightarrow x_0 : A_0$ and built in accordance with the proof search strategy are finite. We take an arbitrary derivation branch $\mathcal{B}$. Since **G3P.CL** rules do not increase the complexity of formulas when going from the conclusion to the premiss(es), the only source of non-termination in the branch is the presence of an infinite number of labels. We need to show that the tree of labels associated to $\mathcal{B}$ is finite. Let us call $\mathcal{T}_{\mathcal{B}}$ the tree associated to $\mathcal{B}$ according to Definition 6.11. This amounts to prove that:

1. Each branch of $\mathcal{T}_{\mathcal{B}}$ has a finite length;
2. Each node of $\mathcal{T}_{\mathcal{B}}$ has a finite number of immediate successors.

Claim 1 is proved by induction on the conditional degree of a label y occurring in the branch. If $d(y) = 0$, y labels either an atomic formula or a propositional formula. In any case, no new world labels are generated from y, and the branch is finite. If $d(y) > 0$, it means that y labels some conditional formula. In this case, y generates at least one world label z, meaning that for some neighbourhood label a, $y \prec_g a$ and $a \prec_g z$. By definition, $y \prec_g a$ if rule $\mathsf{R}>$ or $\mathsf{L}|$ are applied in the derivation branch, introducing formula $a \in N(y)$. Similarly, $a \prec_g z$ if formula $z \in a$ has been introduced in the branch by application of $\mathsf{L}\Vdash^{\exists}$ or $\mathsf{R}\Vdash^{\forall}$. Thus, a new world label z can be generated from a world label y by a combination of the above rules, possibly with the addition of static rules. In any case, it holds that the conditional degree of the formulas labelled with z is strictly smaller than the conditional degree of the formulas labelled with y. To see this, suppose that $y : A > B$ occurs in the consequent of some sequent in the branch. Application of $\mathsf{R}>$ introduces a relational atom $a \in N(y)$, and generates a formula $y \Vdash_a A|B$ in the consequent. Application of rule $\mathsf{R}|$ introduces in the consequent either formula $a \Vdash^{\exists} A$, to which no dynamic rules can be applied, or formula $a \Vdash^{\forall} A \to B$. In this case, rule $\mathsf{R}\Vdash^{\forall}$ can be applied, and a new world label $z \in a$ is generated, along with formula $z : A \to B$ in the consequent. It holds that $d(z) < d(y)$, and similar considerations apply for the other rules combinations. It holds that $d(A_0)$ is bounded by the size of the formula A_0 at the root. Thus, for $n = |A_0|$, the maximal length of each branch of $\mathcal{T}_{\mathcal{B}}$ is bounded by $O(n)$.

Proving claim 2 requires some care. By definition, a world label z is generated by a world label y if there is some neighbourhood label a such that $y \prec_g a$ and $a \prec_g z$, for $k(y) = s$, $k(a) = t$ and $k(z) = u$ with $s < t < u$. To prove that the number of world labels generated by some y is finite, we need to prove that:

a) A world label y generates a finite number of neighbourhood labels;

b) A neighbourhood label a generates a finite number of new world labels.

As for *a*), observe that a new neighbourhood label can be generated only by application of $\mathsf{R}>$ or $\mathsf{L}|$. In the former case, the rule is applied to some formula $y : A > B$ occurring in Δ_{t-1}. Since the formula disappears from Δ_t, rule $\mathsf{R}>$ can be applied only once. Thus, the number of new neighbourhood labels linearly depends on the size of the formula A_0 at the root of the sequent.
The case in which the new neighbourhood is generated by $\mathsf{L}|$ is more complex, since the rule may interact with rule $\mathsf{L}>^{\star}$, as shown in Examples 6.2 and 6.3. To see how the loop is stopped in the general case, suppose that one neighbourhood label $a \in N(y)$ occurs in the antecedent of some sequent in $\mathcal{B}$, along with n conditional formulas $y : A_1 > B_1, \ldots, y : A_n > B_n$. After n applications of $\mathsf{L}>^{\star}$, n formulas $y \Vdash_a A_1|B_1, \ldots, y \Vdash_a A_n|B_n$ occur in the antecedent. By n applications of $\mathsf{L}|$, n new neighbourhood label are generated, along with the following formulas in the antecedent:

$$\begin{array}{c} b_1 \subseteq a, b_1 \in N(y), b_1 \Vdash^{\exists} A_1, b_1 \Vdash^{\forall} A_1 \to B_1 \\ \vdots \\ b_n \subseteq a, b_n \in N(y), b_n \Vdash^{\exists} A_n, b_n \Vdash^{\forall} A_n \to B_n \end{array}$$

Now, rule $\mathsf{L}>^{\star}$ can be applied to all the conditional formulas and all the neighbourhood just introduced. Thus, $n \cdot n$ formulas are generated in the antecedent, along with formulas $b_1 \subseteq b_1, \ldots, b_n \subseteq b_n$ introduced by ref.

$$\begin{array}{ccc} b_1 \subseteq b_1, y \Vdash_{b_1} A_1|B_1, & \ldots & y \Vdash_{b_1} A_n|B_n \\ \vdots & & \vdots \\ b_n \subseteq b_n, y \Vdash_{b_n} A_1|B_1, & \ldots & y \Vdash_{b_n} A_n|B_n \end{array}$$

In principle, application of $\mathsf{L}|$ yields $n \cdot n$ new neighbourhood labels; however, n applications of the rule are blocked by the saturation condition associated to the rule. More precisely, $\mathsf{L}|$ cannot be applied to formula $y \Vdash_{b_1} A_1|B_1$, because formulas $b_1 \subseteq b_1$, $b_1 \in N(x)$, $b_1 \Vdash^{\exists} A_1$ and $b_1 \Vdash^{\forall} A \to B$ occur lower in the branch. Similarly, the saturation condition for $\mathsf{L}|$ blocks applications of the rule to formulas $y \Vdash_{b_2} A_2|B_2$, $y \Vdash_{b_3} A_3|B_3$, and so on. Thus, only $n(n-1)$ new neighbourhood labels are generated. Let $k = n - 1$.

$$\begin{array}{ccc} c_2^1 \subseteq b_1, c_2^1 \Vdash^{\exists} A_2, c_2^1 \Vdash^{\forall} A_2 \to B_2 & \ldots & c_n^1 \subseteq b_1, c_n^1 \Vdash^{\exists} A_n, c_n^1 \Vdash^{\forall} A_n \to B_n \\ \vdots & & \vdots \\ c_1^n \subseteq b_n, c_1^n \Vdash^{\exists} A_1, c_1^n \Vdash^{\forall} A_1 \to B_1 & \ldots & c_k^n \subseteq b_n, b_k^n \Vdash^{\exists} A_k, b_k^n \Vdash^{\forall} A_k \to B_k \end{array}$$

Before applying $\mathsf{L}>^{\star}$, we exhaustively apply the static rules of ref and $\mathsf{Mon}\forall$, obtaining the following formulas:

$$\begin{array}{ccc} c_2^1 \subseteq c_2^1, c_2^1 \Vdash^{\forall} A_1 \to B_1 & \ldots & c_n^1 \subseteq c_n^1, c_n^1 \Vdash^{\forall} A_1 \to B_1 \\ \vdots & & \vdots \\ c_1^n \subseteq c_1^n, c_1^n \Vdash^{\forall} A_n \to B_n & \ldots & c_k^n \subseteq c_k^n, c_k^n \Vdash^{\forall} A_n \to B_n \end{array}$$

We now apply $\mathsf{L} >^\star$, and introduce $n \cdot n(n-1)$ formulas to which $\mathsf{L}|$ can be applied. Let us consider the n formulas generated from application of the rule to label c_2^1. Recall that $\mathsf{L} >^\star$ also introduces local forcing formulas.

$$c_2^1 \Vdash^\exists A_1, \dots, c_2^1 \Vdash^\exists A_n, y \Vdash_{c_2^1} A_1|B_1, y \Vdash_{c_2^1} A_2|B_2, y \Vdash_{c_2^1} A_3|B_3, \dots, y \Vdash_{c_2^1} A_n|B_n$$

The application of $\mathsf{L}|$ to formula $y \Vdash_{c_2^1} A_2|B_2$ is blocked by the saturation condition: formulas $c_2^1 \subseteq c_2^1$, $c_2^1 \Vdash^\exists A_2$ and $c_2^1 \Vdash^\forall A_2 \to B_2$ occur in the branch. Application of the rule to $y \Vdash_{c_2^1} A_1|B_1$ is also blocked: formulas $c_2^1 \Vdash^\exists A_1$ and $c_2^1 \Vdash^\forall A_1 \to B_1$ have been introduced in the branch by $\mathsf{L} >^\star$ and $\mathsf{Mon}\forall$ respectively. Thus, rule $\mathsf{L}|$ can be applied only $n(n-1)(n-2)$ times, generating the same number of new neighbourhood labels. The process continues: after the next applications of $\mathsf{L} >^\star$ and $\mathsf{L}|$, $n(n-1)(n-2)(n-3)$ new labels are introduced, and so on. The number of $\mathsf{L}|$ rule applications blocked by the saturation condition strictly increases, until all the generated neighbourhood labels are blocked.

To be more precise, count as one step in the generation process all applications of $\mathsf{L} >^\star$, $\mathsf{Mon}\forall$, ref and $\mathsf{L}|$ to a sequent. During the i-th step, rule $\mathsf{L} >^\star$ generates a number n of formulas $x \Vdash_e G|H$ for each neighbourhood label occurring in the sequent. Then, rule $\mathsf{L}|$ can be applied, introducing a new neighbourhood label for each application. However, out of every n formulas $x \Vdash_e G|H$, $i-1$ applications of $\mathsf{L}|$ are blocked.

$$\textit{\# of new neighbourhood labels at the } i^{th} \textit{ step} \;=\; \frac{n!}{(n-i)!}$$

It follows that after $n+1$ steps, all the generated neighbourhood labels are blocked and, as a consequence, all applications of $\mathsf{L} >^\star$ and $\mathsf{L}|$ are blocked. In general, for each neighbourhood label $a \in N(y)$ and n conditional formulas labelled with y, we generate at most

$$\sum_{k=1}^{n-1} \frac{n!}{(n-k)!}$$

new neighbourhood labels. The number of neighbourhood labels generated by $\mathsf{L}|$ and $\mathsf{L} >$ is bounded by $O((n-2) \cdot n!)$, since at most $n-2$ terms appear in the sum of labels, and the biggest term in the sum is $n!$. This can be approximated to $O(n^2 \cdot n!)$.

To prove b), recall that a new world label z is generated from a neighbourhood label a if rule $\mathsf{L} \Vdash^\exists$, $\mathsf{R} \Vdash^\forall$, or $\mathsf{L}|$ are applied in the derivation. Since in all these rules the principal formula disappears from the premiss, each rule can be applied at most once to each suitable formula, generating one world label for each application. Thus, the number of world labels generated linearly depends on the size of the formula A_0 at the root of the derivation.

Since $\mathcal{T}_{\mathcal{B}}$ has a finite number of nodes, the world and neighbourhood labels in the derivation are finite. Since the pure formulas are in a finite number (all subformulas of A_0), in a finite number of steps proof proof search terminates, yielding either a saturated sequent or an initial sequent. $\square$

Take $n = |A_0|$. The number of labels generated from a node of $\mathcal{T}_{\mathcal{B}}$ is counted as follows. The number of neighbourhood labels generated by $\mathsf{R}>$ is $O(n)$. Since the number of conditional formulas in the derivation is bounded by $|A_0|$, the number of neighbourhood labels generated by $\mathsf{L}|$ and $\mathsf{L}>^*$ is bounded by $O(n^2 \cdot n!)$. Each neighbourhood label generates one new world labels; thus, the maximal number of world labels generated from a world label is bounded by $O(n^2 \cdot n!)$. To conclude, since the maximal length of each branch is bounded by $O(n)$, the maximal number of world labels introduced in a derivation branch is bounded by $O(n^3 \cdot n!)$. To obtain a complexity bound for the decision procedure associated to **G3P.CL**, the maximal number of labels has to be combined with the number of formulas generated at each step. The exponential bound on labels, however, already shows that the complexity of the decision procedure is NEXPTIME, far from the PSPACE bound known for $\mathbb{PCL}$ (see Remark 6.4).

6.2 Decidability for extensions

Theorem 6.14 can be extended to the calculi for most extensions of $\mathbb{PCL}$.

We show how sequent calculi for logics with normality, total reflexivity, weak centering, centering and uniformity terminate. We do not treat extensions of **G3P.CL** with the rules for absoluteness. In these logics all $N(x)$ are the same, and there is no need to keep track of the system of neighbourhood $N(x)$ to which a certain neighbourhood α belongs. This simplification is not reflected by the sequent calculus **G3P.A**, which is instead defined as a *modular* extension of **G3P.CL**. Thus, proving termination of **G3P.A** is not worth, since the simplest extension of $\mathbb{PCL}$ would result in having the most complex decision procedure[16].

In order to treat the extensions of $\mathbb{PCL}$ we define saturation conditions for the additional rules and prove that the tree of labels corresponding to a derivation branch is finite.

The rules we are concerned with are N, $\mathsf{0}$, T, W, C, single, repl_1, repl_2, U_1 and U_2. Proof of termination for sequent calculi displaying a combination of these rules can be obtained by combining the proof strategies exposed in this section. We start by adding to the conditions in Figure 6 the saturation conditions for these new rules, listed in Figure 7.

Definition 6.15. The proof search strategy from Definition 6.9 is supplemented with the following clause:

3. Rule $\mathsf{0}$ can be applied to a sequent and a formula $a \in N(x)$ only if some formula $a \Vdash^{\exists} A$ occurs in the consequent, or some formula $a \Vdash^{\forall} A$ occurs in the antecedent.

[16] Refer to [10] for terminating a labelled sequent calculus more suitable to treat the condition of absoluteness. The resulting decision procedure, however, is still far from optimal.

0	If $a \in N(x)$ is in Γ then $y \in a$ is in Γ for some y
N	If x is in $\downarrow \Gamma \cup \downarrow \Delta$ then for some a, $a \in N(x)$ is in Γ
T	If x is in $\downarrow \Gamma \cup \downarrow \Delta$, there is an a such that $a \in N(x)$ and $x \in a$ are in Γ
W	If $a \in N(x)$ is in Γ then $x \in a$ is in Γ
C	If $a \in N(x)$ is in Γ, both $\{x\} \in N(x)$ and $\{x\} \subseteq a$ are in Γ
single	If $\{x\} \in N(x)$ is in Γ, then $x \in \{x\}$ is in Γ
repl_1	If $y \in \{x\}$ is in Γ, and if some formula $At(x)$ is in Γ, then $At(y)$ is in Γ
repl_2	If $y \in \{x\}$ is in Γ, and if some formula $At(y)$ is in Γ, then $At(x)$ is in Γ
U_1	If $a \in N(x)$, $y \in a$, $b \in N(y)$ and $z \in b$ are in Γ, then for some c, $c \in N(x)$ and $z \in c$ are in Γ
U_2	If $a \in N(x)$, $y \in a$, $b \in N(x)$ and $z \in b$ are in Γ, then for some c, $c \in N(y)$ and $z \in c$ are in Γ

Figure 7: Saturation conditions for extensions

Let us see how the proof search strategy stops the two new cases of loop generated by the rules for extensions. Interaction of 0 and N generates the following loop.

$$
\cfrac{\cfrac{\cfrac{\cfrac{\vdots}{z \in b, b \in N(y), y \in a, a \in N(x), x : A, \Gamma \Rightarrow \Delta}}{b \in N(y), y \in a, a \in N(x), x : A, \Gamma \Rightarrow \Delta}\,\mathsf{0}}{y \in a, a \in N(x), x : A, \Gamma \Rightarrow \Delta}\,\mathsf{N}}{\cfrac{a \in N(x), x : A, \Gamma \Rightarrow \Delta}{x : A, \Gamma \Rightarrow \Delta}\,\mathsf{N}}\,\mathsf{0}
$$

If no formulas $a \Vdash^{\exists} A$ occur in Δ and no formulas $a \Vdash^{\forall} A$ occur in Γ, the first application of rule 0 is blocked. Suppose $a \Vdash^{\exists} A$ occurs in Δ. Then 0 is applied, but if restriction 3 is not met by neighbourhood b, the second uppermost application of 0 is stopped. The number of formulas $a \Vdash^{\exists} A$ in the consequent and $a \Vdash^{\forall} A$ in the antecedent is bounded by the conditional degree of formulas at the root; thus, the loop is stopped. Intuitively, rule 0 needs to be applied only to the neighbourhood label introduced by N, to ensure that it is not empty[17]. The neighbourhoods introduced by T, U_1 or U_2 already contain at least one element, and no loops with 0 arise.

Applications of U_1 and U_2 generate the following loop, where we take $\Omega =$

[17] Refer to the derivation of axiom (N) in the Appendix.

$a \in N(x), y \in a, b \in N(y), z \in b.$

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots \atop f \in N(y), z \in f, e \in N(x), z \in e, d \in N(y), z \in d, c \in N(y), z \in c, \Omega, \Gamma \Rightarrow \Delta}{e \in N(x), z \in e, d \in N(y), z \in d, c \in N(y), z \in c, \Omega, \Gamma \Rightarrow \Delta}\ \mathsf{U}_2}{d \in N(y), z \in d, c \in N(y), z \in c, \Omega, \Gamma \Rightarrow \Delta}\ \mathsf{U}_1}{c \in N(x), z \in c, \Omega, \Gamma \Rightarrow \Delta}\ \mathsf{U}_2}{\Omega, \Gamma \Rightarrow \Delta}\ \mathsf{U}_1$$

The saturation condition for U_2 blocks the first bottom-up application of the rule: there is a neighbourhood label b such that $b \in N(y)$ and $z \in d$ are in Γ. Similarly, a loop generated by repl_1 and repl_2 is blocked by their saturation conditions.

We now prove termination for the sequent calculi extending **G3P.CL**, adapting the proof of termination for **G3P.CL** (Theorem 6.14). Observe that Lemma 6.12 holds for all the extensions considered: thus, the world labels occurring in a derivation branch form a tree according to the relation $\prec$.

Theorem 6.16 (Termination)**.** *Let* **K** *one of* **N**, **T**, **W**, **C**, **U**, **NU**, **TU**, **WU**, **CU**. *Root-first proof search for a sequent* $\Rightarrow x_0 : A_0$ *in sequent calculi* **G3P.K***, built in accordance with the strategy, terminates in a finite number of steps, with each leaf of the derivation tree containing either an initial sequent or a saturated sequent.*

Proof. As in the proof of Theorem 6.14, we need to check that the tree of labels $\mathcal{T}_\mathcal{B}$ associated to an arbitrary derivation branch is finite:

1. Each branch of $\mathcal{T}_\mathcal{B}$ has a finite length;

2. Each node of $\mathcal{T}_\mathcal{B}$ has a finite number of immediate successors.

As for 1, the proof remains the same as in Theorem 6.14. Rule $\mathsf{0}$ introduces a new world label but, as we have seen, applications of this rule are restricted: the rule can be applied only if afterwards some rule of local forcing can be applied to the new world label. Since the number of local forcing formulas occurring in a derivation branch is bounded by the size of formula A_0, the length of a branch in $\mathcal{T}_\mathcal{B}$ starting from a world label y is still bounded by $O(n)$, for $n = |A_0|$. Rules T, W and single introduce in derivation branch a world label x generated by x itself. By definition, in order to have $x \prec y$ we need that $x \neq y$; and thus, the rules do not introduce a new node in the tree of labels. Rule C does not introduce a new world label in the derivation. Replacement rules are applied only to atomic formulas, and operate exclusively on world labels which are already present in the derivation. Similarly, rules U_1 and U_2 do not introduce new world labels in the derivation; thus, they do not affect the length of a branch in $\mathcal{T}_\mathcal{B}$.

The proof of 2 remains basically the same as in Theorem 6.14, meaning that the count of the number of new neighbourhood labels generated by one neighbourhood label and n formulas $y : A_1 > B_1 \ldots, y : A_n > B_n$ is still bounded by $O(n^2 \cdot n!)$, for $n = |A_0|$. However, the number of neighbourhood labels generated from a world label increases, due to the presence of rules for extensions. Each rule N, T, C, if applicable, adds at most one neighbourhood label $a \in N(y)$ or $\{y\} \in N(y)$ to a world label y. The number of additional neighbourhood labels generated by each world label is bounded by the size of the formula A_0 at the root; in logics with normality, total reflexivity and centering this number is at most $O(n + 3)$.

As for the rules of replacement, given a world label y and a formula $z \in \{y\}$, these rules rules may introduce in the derivation formulas $a \in N(z)$ or $a \in N(y)$. Thus, the number of additional world labels introduced from a world label y is bounded by the size of formula A_0, and thus by $O(n)$ (as before), to which we have to add the number of applications of replacement rules introducing relational atoms $a \in N(y)$. Since replacement rules can be applied at most once to each $z \in \{y\}$ and $a \in N(z)$, the total number of additional neighbourhood labels is bounded by $O(2n)$.

A similar reasoning holds for U_1 and U_2. The saturation conditions for uniformity prevent the application of both U_1 and U_2 to formulas $c \in N(x)$ (or $c \in N(y)$), and $z \in c$ if the neighbourhood label has been generated by the rules of uniformity. Thus, only one rule of uniformity (U_1 or U_2) can be applied out of every 4 relational atoms $a \in N(x)$, $y \in a$, $b \in N(y)$ (or $b \in N(x)$) and $z \in b$. Moreover, the rule can be applied at most once to these labels. We can thus estimate the maximal number of additional neighbourhood labels introduced by a world label to be bounded by $O(2n)$.

Following the same reasoning as for **G3P.CL**, we have that the maximal number of world labels generated from a world label for calculi without centering or uniformity is given by $O(n^2 \cdot n!)$, while for calculi with centering or uniformity the bound is $O(2n \cdot n \cdot n!)$. □

To conclude, the maximal length of each branch in $\mathcal{T}_{\mathcal{B}}$ combined with the maximal number of nodes generated from a node yields the following maximal bounds for world labels introduced in a derivation branch: $O(n^3 \cdot n!)$ in case of calculi without centering or uniformity, and $O(2n \cdot n^2 \cdot n!)$ for calculi with centering or uniformity, always taking $n = |A_0|$. In both cases, the decision procedure associated to the logic is NEXPTIME.

7 Semantic completeness

Completeness of a sequent calculus can be proved either with respect to the axiom system or with respect to the class of models for the logic. Theorem 5.9, along with Theorem 5.8 of cut-admissibility ensures completeness of all calculi with respect to the axiomatization of the corresponding logics. In this section we prove completeness of the calculi with respect to classes of models (also called *semantic completeness*). We show that if a formula is valid in the class of neighbourhood models for a given logic, then it is derivable in the corresponding labelled calculus. As usual we prove the counterpositive statement: if a formula is not derivable in a certain proof system, we can construct a countermodel for the formula in the corresponding class of models. The model will be extracted from a saturated sequent occurring as a leaf of the failed derivation tree for that formula.

Since the proof requires to build a countermodel from a saturated sequent, termination of the calculi is needed. For this reason, we prove semantic completeness of all the systems for which we proved termination, that is, all systems except for those with the condition of absoluteness.

7.1 Completeness for G3P.CL

Theorem 7.1. *Let $F \in \mathcal{L}$. If F is valid in $\mathcal{M}_{\mathcal{C}l}$, in symbols $\vDash_{\mathcal{C}l} F$, then $\vdash_{\mathsf{G3P.CL}} \Rightarrow x : F$.*

Proof. We prove the counterpositive: if sequent $\Rightarrow x : F$ is not derivable in **G3P.CL**, then F is not valid in $\mathcal{M}_{\mathcal{C}l}$. Suppose we have a failed **G3P.CL** derivation of $\Rightarrow x : F$, and let $\Gamma \Rightarrow \Delta$ be a saturated sequent occurring as a leaf in the branch. We shall construct a *finite* neighbourhood model $\mathcal{M}_{\mathcal{B}} \in \mathcal{M}_{\mathcal{C}l}$ that satisfies all formulas in $\downarrow \Gamma$ and falsifies all formulas in $\downarrow \Delta$. As a consequence, $\mathcal{M}_{\mathcal{B}}$ falsifies $x : F$ and, since there exists a model and a world that falsify F, we conclude that F is not valid in $\mathcal{M}_{\mathcal{C}l}$.

The countermodel $\mathcal{M}_{\mathcal{B}}$ contains the semantic informations encoded in the sequents of the derivation branch. Let

$$S_{\mathcal{B}} = \{x \mid x \in (\downarrow \Gamma \cup \downarrow \Delta)\} \qquad N_{\mathcal{B}} = \{a \mid a \in (\downarrow \Gamma \cup \downarrow \Delta)\}$$

Then, we associate to each $a \in N_{\mathcal{B}}$ a neighbourhood as follows:

$$\alpha_a = \{y \in S_{\mathcal{B}} \mid y \in a \textit{ belongs to } \Gamma\}$$

Thus, for each neighbourhood a, $\alpha_a \subseteq S_{\mathcal{B}}$. We construct the neighbourhood model $\mathcal{M}_{\mathcal{B}} = \langle W_{\mathcal{B}}, N_{\mathcal{B}}, [\![\ ]\!]_{\mathcal{B}} \rangle$ as follows.

- $W_{\mathcal{B}} = S_{\mathcal{B}}$
- For any $x \in W_{\mathcal{B}}$, $N_{\mathcal{B}}(x) = \{\alpha_a \mid a \in N(x) \textit{ belongs to } \downarrow \Gamma\}$
- For p atomic, $[\![p]\!]_{\mathcal{B}} = \{x \in W_{\mathcal{B}} \mid x : p \textit{ belongs to } \downarrow \Gamma\}$

We now show that $\mathcal{M}_{\mathcal{B}} = \langle W_{\mathcal{B}}, N_{\mathcal{B}}, [\![\ ]\!]_{\mathcal{B}} \rangle$ satisfies the property of non-emptiness of neighbourhood models for $\mathbb{PCL}$: we have to verify that every $\alpha_a \in N(x)$ contains at least one element. If $a \in N(x)$ occurs in the sequent, it must have been introduced either by $\mathsf{R}>$ or $\mathsf{L}|$. By the saturation conditions associated to both rules it holds that $a \Vdash^{\exists} C$ occurs in $\downarrow \Gamma$. Thus, by the saturation condition $\mathsf{R} \Vdash^{\exists}$, formula $y \in a$ occurs in Γ.
Moreover, the model $\mathcal{M}_{\mathcal{B}}$ satisfies the following property:

($*$) If $a \subseteq b$ belongs to Γ, then $\alpha_a \subseteq \alpha_b$

To verify $(*)$, suppose $y \in \alpha_a$. This means that $y \in a$ belongs to Γ; then, by the saturation condition $\mathsf{L} \subseteq$, also $y \in b$ belongs to Γ. By definition of the model we have $y \in \alpha_b$, and thus that $\alpha_a \subseteq \alpha_b$.

Next, define a realization (ρ, σ) such that $\rho(x) = x$ and $\sigma(a) = \alpha_a$ and prove the following claims:

[**Claim 1**] If $\mathcal{F}$ is in $\downarrow \Gamma$, then $\mathcal{M}_{\mathcal{B}} \vDash_{\rho,\sigma} \mathcal{F}$;
[**Claim 2**] If $\mathcal{F}$ is in $\downarrow \Delta$, then $\mathcal{M}_{\mathcal{B}} \nvDash_{\rho,\sigma} \mathcal{F}$;

The two claims are proved by induction on the weight of the formula $\mathcal{F}$.

[**a**] If $\mathcal{F}$ is a formula of the form $a \in N(x)$, $x \in a$ or $a \subseteq b$, Claim 1 holds by definition of $\mathcal{M}_{\mathcal{B}}$, and Claim 2 is empty. For the case of $a \subseteq b$, employ the fact $(*)$ above.

[**b**] If A is a labelled atomic formula $x : p$, the claims hold by definition of the model; by the saturation condition associated to init no inconsistencies arise. If $A \equiv \bot$, the formula is not forced in any model and Claim 2 holds, while Claim 1 holds by the saturation clause associated to $\bot_{\mathsf{L}}$. If A is a conjunction, disjunction or implication, both claims hold for the corresponding saturation conditions and by inductive hypothesis on formulas on smaller weight.

[**c**] If $A \equiv a \Vdash^{\exists} A$ is in $\downarrow \Gamma$, then by the saturation clause associated to $\mathsf{L} \Vdash^{\exists}$ for some x there are $x \in a$, $x : A$ are in $\downarrow \Gamma$. By definition of the model $\mathcal{M}_{\mathcal{B}}$, for some x, $x \in \alpha_a$. Then, since $w(x : A) < w(a \Vdash^{\exists} A)$, apply the inductive hypothesis and obtain $\mathcal{M}_{\mathcal{B}} \vDash x : A$. Therefore, by definition of satisfiability, $\mathcal{M}_{\mathcal{B}} \vDash \alpha_a \Vdash^{\exists} A$.
If $a \Vdash^{\exists} A$ is in $\downarrow \Delta$, then it is also in Δ. Consider an arbitrary world x in α_a. By definition of $\mathcal{M}_{\mathcal{B}}$ we have that $x \in a$ is in Γ; apply the saturation condition associated to $\mathsf{R} \Vdash^{\forall}$ and obtain that $x : A$ is in $\downarrow \Delta$. By inductive hypothesis, $\mathcal{M}_{\mathcal{B}} \nvDash x : A$; thus, since this line of reasoning holds for arbitrary x, we can conclude by definition of satisfiability that $\mathcal{M}_{\mathcal{B}} \nvDash \alpha_a \Vdash^{\exists} A$. The case in which $A \equiv a \Vdash^{\forall} A$ is similar.

[**d**] If $x \Vdash_a A|B$ is in $\downarrow \Gamma$, then by the saturation condition associated to $\mathsf{L}|$ for some c it holds that $c \in N(x)$ and $c \subseteq a$ are in Γ, and $a \Vdash^{\exists} A$, $a \Vdash^{\forall} A \to B$ are in $\downarrow \Gamma$. By definition of the model, $\alpha_c \subseteq \alpha_a$, and by inductive hypothesis $\mathcal{M}_{\mathcal{B}} \vDash \alpha_c \Vdash^{\exists} A$ and $\mathcal{M}_{\mathcal{B}} \vDash \alpha_c \Vdash^{\forall} A \to B$. By definition, this yields $\mathcal{M}_{\mathcal{B}} \vDash x \Vdash_a A|B$.
If $x \Vdash_a A|B$ is in $\downarrow \Delta$, consider a neighbourhood $\alpha_c \subseteq \alpha_a$ in $N(x)$. Then by definition of $\mathcal{M}_{\mathcal{B}}$ we have that $c \in N(x)$ and $c \subseteq a$ are in Γ; apply the saturation condition associated to $\mathsf{R}|$ and obtain that either $c \Vdash^{\exists} A$ or $c \Vdash^{\forall} A \to B$ is in $\downarrow \Delta$. By inductive hypothesis, either $\mathcal{M} \nvDash \alpha_c \Vdash^{\exists} A$ or $\mathcal{M}_{\mathcal{B}} \nvDash \alpha_c \Vdash^{\forall} A \to B$. In both cases, by definition $\mathcal{M}_{\mathcal{B}} \nvDash x \Vdash_a A|B$.

[**e**] If $x : A > B$ is in $\downarrow \Gamma$, then it is also in Γ. Consider an arbitrary neighbourhood α_a in $N(x)$. By definition of $\mathcal{M}_{\mathcal{B}}$ we have that $a \in N(x)$ is in Γ; apply the saturation condition associated to $\mathsf{L} >'$ and conclude that either $a \Vdash^{\exists} A$ is in $\downarrow \Delta$, or $x \Vdash_a A|B$ is in $\downarrow \Gamma$. By inductive hypothesis, it holds that either $\mathcal{M}_{\mathcal{B}} \nvDash \alpha_a \Vdash^{\exists} A$ or $\mathcal{M}_{\mathcal{B}} \vDash x \Vdash_a A|B$. In both cases, by definition $\mathcal{M}_{\mathcal{B}} \vDash x : A > B$.

If $x : A > B$ is in $\downarrow \Delta$, by the saturation condition associated to $\mathsf{R}>$, for some a it holds that $a \in N(x)$ is in Γ, $a \Vdash^{\exists} A$ is in $\downarrow \Gamma$ and $x \Vdash_a A|B$ is in $\downarrow \Delta$. By inductive hypothesis, $\mathcal{M}_{\mathcal{B}} \vDash \alpha_a \Vdash^{\exists} A$ and $\mathcal{M}_{\mathcal{B}} \nvDash x \Vdash_a A|B$, thus, by definition, we have $\mathcal{M}_{\mathcal{B}} \nvDash x : A > B$.
The cases of $x : A \wedge B$, $x : A \vee B$ and $x : A \to B$ are proved in a similar way, but are simpler, since the truth condition of these operators takes into account only one world, x. □

Theorem 7.1 together with the soundness of **G3P.CL** provides a constructive proof of the *finite model property* for the logic: if A is satisfiable in a model (meaning that $\neg A$ is not valid), by soundness of the calculi $x : \neg A$ is not provable. Thus by Theorem 7.1 we build a finite countermodel of $\neg A$, that is a finite model in which A is satisfiable. The same holds for the calculi for extensions of $\mathbb{PCL}$, once their semantic completeness has been proved.

7.2 Semantic completeness for extensions

Semantic completeness for sequent calculi with normality, total reflexivity, weak centering and uniformity can be proved similarly as for **G3P.CL**. Extensions of the calculi with the rule for centering require a modification on the countermodel construction, to account for singleton neighbourhoods. To obtain a proof for calculi that extend **G3P.CL** with more than one rule for extension (i.e. **G3P.NU**), it suffices to combine the proof strategies for each case.

Theorem 7.2. *Let $F \in \mathcal{L}$ and* **K** *be one of* **N**, **T**, **W**, **U**, **NU**, **TU**, **WU**. *If F is valid in $\mathcal{M}_{\mathcal{K}}$, then $\vdash_{\mathsf{G3P.K}} F$.*

Proof. The proof proceeds as the one of Theorem 7.1. For the case of normality, a clause is added in the countermodel construction; however, Claim 1 and 2 continue to hold in the model. For the remaining cases, the countermodel construction does not change, and it only remains to verify that the countermodel $\mathcal{M}_{\mathcal{B}}$ satisfies the properties of normality, total reflexivity, weak centering and uniformity, provided that the corresponding rules and saturation conditions are added to the calculus.

Normality: To construct a countermodel for logics featuring *only* normality, the following case distinction applies, for $Q = \forall, \exists$:

- If $a \in N(x)$ occurs in Γ, and there are some formulas $a \Vdash^{Q} A$ in $\downarrow \Gamma \cup \downarrow \Delta$, the countermodel $\mathcal{M}_{\mathcal{B}}$ is defined as in the case of $\mathbb{PCL}$;
- If $a \in N(x)$ occurs in Γ, but no formulas $a \Vdash^{Q} A$ occur in $\downarrow \Gamma \cup \downarrow \Delta$, we set: $W_{\mathcal{B}} = S_{\mathcal{B}} \cup \{u\}$, for some variable u not occurring in Γ; $\alpha_a = \{u\}$ and $N_{\mathcal{B}}(u) = \{\{u\}\}$.

The model satisfies the condition of normality: according to the saturation condition N, for every x occurring in $\downarrow \Gamma$, there is a such that $a \in N(x)$ occurs in Γ. By definition of $\mathcal{M}_{\mathcal{B}}$, $\alpha_a \in N_{\mathcal{B}}(x)$. Moreover, we have to verify that nonemptiness of the model holds also for the neighbourhood α_a introduced by the rule. If there are some formulas $a \Vdash^{Q} A$ occurring in $\downarrow \Gamma \cup \downarrow \Delta$, the saturation condition associated to either $\mathsf{0}$, $\mathsf{L} \Vdash^{\exists}$ or $\mathsf{R} \Vdash^{\forall}$ ensures that there is at least one formula $y \in a$ in Γ. If there are no such formulas, the application of N is not relevant to the derivation; following the definition, we introduce an arbitrary world u to be placed in the neighbourhood[18].

Total reflexivity: According to the saturation condition T, for every x occurring in $\downarrow\Gamma \cup \downarrow\Delta$ also $a \in N(x)$, $x \in a$ occur in Γ. By definition of $\mathcal{M}_{\mathcal{B}}$, this means that $\alpha_a \in N(x)$ and $x \in \alpha_a$, and total reflexivity holds.

Weak centering: Suppose $\alpha_a \in N(x)$. We want to show that $x \in \alpha_a$. By definition, if $\alpha_a \in N(x)$ then $a \in N(x)$ occurs in Γ. By the saturation condition associated to W, it holds that also $x \in a$ occurs in Γ; thus, by definition of the model $x \in \alpha_a$.

Uniformity: Suppose $y \in \bigcup N(x)$, which means that $y \in \alpha_a$ and $\alpha_a \in N(x)$. By definition, $a \in N(x)$ and $y \in a$ occur in Γ. We have to show that $\bigcup N(x) = \bigcup N(y)$, that is:

$$z \in \bigcup N(x) \text{ iff } z \in \bigcup N(y)$$

Assume $z \in \bigcup N(x)$. This means that $z \in \alpha_b$ and $b \in N(x)$ and, by definition, $z \in b$ and $b \in N(x)$ occur in Γ. By the saturation condition associated to $\mathsf{U_2}$, we have that for some c, $c \in N(y)$ and $z \in c$ occur in Γ. Thus, $z \in \alpha_c$ and $\alpha_c \in N(y)$, meaning that $z \in \bigcup N(y)$. The saturation condition associated to $\mathsf{U_1}$ is needed to prove the other direction. □

Theorem 7.3. *Let $F \in \mathcal{L}$ and* **K** *be* **C** *or* **CU**. *If F is valid in $\mathcal{M}_{\mathcal{K}}$, then $\vdash_{\mathsf{G3P.K}} F$.*

Proof. In this case, worlds of the countermodel are not defined as the set $S_{\mathcal{B}}$ of labels occurring in the branch, but as *equivalence classes* $[x]$ with respect to the relation $y \in \{x\}$, which we will show to be an equivalence relation. Then, we require $[x]$ to be contained in any neighbourhood of $N(x)$. For $S_{\mathcal{B}}$, $N_{\mathcal{B}}$ and α_a as defined before, let

$$[x] = \{y \in S_{\mathcal{B}} \mid y \in \{x\} \textit{ occurs in } \Gamma\};$$
$$[x] \subseteq \alpha_a, \text{ for } a \in N(x) \text{ occurring in } \Gamma.$$

We construct a model $\mathcal{M}^c_{\mathcal{B}} = \langle W^c, N^c, [\![\ ]\!]^c \rangle$ as follows:

- $W^c = \{[x] \mid x \in S_{\mathcal{B}}\}$;
- for each $[x] \in W^c$, $N^c([x]) = \{\alpha_a \mid a \in N(x) \textit{ belongs to } \downarrow\Gamma\}$;
- for p atomic, $[\![p]\!]^c = \{[x] \in W^c \mid x : p \textit{ belongs to } \downarrow\Gamma\}$.

We first prove that $y \in \{x\}$ is an equivalence relation. The relation is *reflexive*: for each x occurring in Γ, $x \in \{x\}$ occurs in Γ. This holds from the saturation conditions associated to N, C and single. To prove that the relation is *symmetric*, we have to show that if $y \in \{x\}$ occurs in Γ, then also $x \in \{y\}$ occurs in Γ. By reflexivity, we have that $y \in \{y\}$. Thus, by the saturation condition

[18] There is no need to verify non-emptiness for stronger conditions of total reflexivity and weak centering, since the rules added to the calculus add a world belonging to the neighbourhood introduced.

associated to repl_2, we have that also $x \in \{y\}$ belongs to Γ. To prove the converse, use saturation condition associated to repl_1. To show that the relation is *transitive* we have to prove that if $y \in \{x\}$ and $x \in \{z\}$ occur in Γ, also $y \in \{z\}$ occurs in Γ. By saturation conditions N and C and single, we have that also $\{z\} \in N(z)$ occurs in Γ. By the saturation condition associated to repl_1 applied to $x \in \{z\}$, also $\{z\} \in N(x)$ occurs in the sequent; thus, by the saturation condition associated to C we have that both formulas $\{x\} \subseteq \{z\}$ and $\{x\} \in N(z)$ occur in Γ. Finally, by the saturation condition associated to $\mathsf{L}\subseteq$, since $y \in \{x\}$ and $\{x\} \subseteq \{z\}$, we have that $y \in \{z\}$ occurs in the sequent.

Next we need to show that the definitions of $N^c([x])$ and $[\![p]\!]^c$ do not depend on the chosen representative of the equivalence class in question.

i) if $y \in [x]$, then $a \in N(x)$ is in Γ if and only if $a \in N(y)$ is in Γ;
ii) if $y \in [x]$, then $x : p$ is in Γ if and only if $y : p$ is in Γ.

Fact *i)* follows from the saturation conditions associated to repl_1 and repl_2, applied to on the formulas $a \in N(x)$ and $a \in N(y)$. Fact *ii)* follows from application of the same saturation conditions to $x : p$ and $y : p$.

The model $\mathcal{M}^c_{\mathcal{B}}$ satisfies the property of centering. Observe that in our model $\{x\}$ corresponds to $[x]$: both are defined as the set containing exactly one element, x. Suppose $\alpha_a \in N(x)$; we have to show that $\{x\} \subseteq \alpha_a$ and $\{x\} \in N(x)$. By definition of the model we have that $[x] \subseteq \alpha_a$, and from this and $\alpha_a \in N(x)$ it follows that $[x] \in N([x])$; thus, strong centering holds.
The following facts are needed in the proof Claims 1 and 2 below.

1) If $a \subseteq b$ belongs to Γ, then $\alpha_a \subseteq \alpha_b$;
2) if $[x] \in [\![A]\!]$ and $y \in [x]$, then $[y] \in [\![A]\!]$;
3) If $[x] \in [\![A]\!]$, then $x : A$ belongs to $\downarrow \Gamma$.

Fact 1) is proved in the same way as $(*)$ of the proof of Theorem 7.1; the proofs of 2) and 3) are immediate from admissibility of repl_1 and repl_2 in their generalized form (Lemma 5.10).

Finally, we define define a realization (ρ, σ) such that $\rho(x) = [x]$ and $\sigma(a) = \alpha_a$, and prove that:

[Claim 1] If $\mathcal{F}$ is in $\downarrow \Gamma$, then $\mathcal{M}^c_{\mathcal{B}} \vDash_{\rho,\sigma} \mathcal{F}$;
[Claim 2] If $\mathcal{F}$ is in $\downarrow \Delta$, then $\mathcal{M}^c_{\mathcal{B}} \nvDash_{\rho,\sigma} \mathcal{F}$.

Again, $\mathcal{F}$ denotes the labelled formulas of the language, including $y \in \{x\}$, $\{x\} \in N(x)$, $\{x\} \in a$. The two cases are proved by distinction of cases, and by induction on the height of the derivation. If $\mathcal{F}$ is a relational formula that does not contain any singleton, Claim 1 holds by definition of the model, and Claim 2 is empty as in case a) of proof of the previous models. Similarly, if $\mathcal{F}$ is either $y \in \{x\}$, $\{x\} \in N(x)$ or $\{x\} \subseteq a$, Claim 1 is satisfied by definition.

The cases b)- e) of the previous proof remain unchanged; condition 2) ensures that all the elements of an equivalence class of world labels satisfy the same sets of formulas. $\square$

8 Related works and conclusions

8.1 Discussion of related works

Semantic issues

Concerning the semantics, a few works have considered neighbourhood models for $\mathbb{PCL}$ or closely related logics. The relation between neighbourhood models and preferential models is based on a well-known duality between partial orders and what are known as Alexandrov topologies (refer, for instance, to [21]). According to this result, neighbourhood models are built by associating to each world a topology in which the neighbourhoods are the open sets. For conditional logics this duality is studied in detail in [17]. However, the topological semantics of [17] imposes closure under arbitrary unions and non-empty intersections on the neighbourhoods. These conditions are not required by the logic, as shown by the Completeness Theorem 3.15, and we have not assumed them in the definition of neighbourhood models.

A kind of neighbourhood semantics, called Broccoli semantics, has been considered in [9]. A *broccoli model* is defined as a triple $\langle W, \{\mathcal{B}_x\}_{x\in W}, [\![\ ]\!]\rangle$, where $\{\mathcal{B}_x\}_{x\in W}$ is a *broccoli flower*, i.e., a neighbourhood, satisfying a kind of limit condition. In [9], it is shown that the logic BL characterised by the Broccoli Semantics coincides with $\mathbb{PCL}$. Completeness of BL is obtained through Burgess' result.

Yet another kind of neighbourhood semantics bearing some similarity to ours is the Premise semantics, considered in the seminal work by Veltman [29]. Premise semantics is shown equivalent to preferential semantics (called "ordered semantics"). Premise models are neighbourhood models which do not require any additional properties, as in our definition. However, the definition of the conditional is different from ours, as it considers arbitrary intersections of neighbourhoods. Then, the result of *strong* completeness is proved indirectly by resorting to preferential semantics (whence generalising Burgess' result).

In this respect, the direct completeness result with respect to the neighbourhood semantics contained in this work is new. In future work we wish to complete it with the relevant missing cases of $\mathbb{PW}$ and $\mathbb{PWU}$.

Proof systems

Concerning proof systems, very few calculi are known for $\mathbb{PCL}$ and its extensions. In [8] the authors propose labelled tableaux calculi for $\mathbb{PCL}$ and its extensions, covering all the logics considered here, including the extensions of $\mathbb{PCL}$ with nesting, i. e., Lewis' logics. The calculi are based on preferential semantics with the Limit assumption, and are defined by extending the language by pseudo-modalities indexed on worlds. More precisely, the tableaux calculi make use of formulas of the form $z <_x y$ to denote the preferential relation associated to a world x, which can be read as "z is more similar to x than y", and formulas of the form $y : \Box_x A$ meaning that A holds in all the worlds z accessible from x and such that $z <_x y$.

The tableaux calculi from [8] cover all logics considered in this work, but they are inherently different from the ones we introduced, due to the presence of the Limit assumption. As a difference with the presxent work, termination is obtained by relatively complex blocking conditions.

As a side note, the neighbourhood semantics could be reformulated by assuming the Limit assumption as follows. Given a formula A, let $\mathcal{N}^A(x)$ be the set of neighbourhoods $\alpha \in N(x)$ minimal with respect to set inclusion and such that $\alpha \Vdash^{\exists} A$. In this setting, the Limit assumption says that $\mathcal{N}^A(x)$ is non-empty whenever $\bigcup N(x)$ contains some A-world. In presence of the assumption, we could reformulate the truth condition of a conditional as follows: $A > B$ is forced at x if each neighbourhood α in $\mathcal{N}^A(x)$ universally forces $A \to B$. Corresponding calculi could possibly be developed based on this semantics.

Labelled sequent calculi based on preferential semantics for $\mathbb{PCL}$ and its extensions, including counterfactual logics, are presented in [13]. In this case, the semantics is defined *without* the Limit assumption. From a proof-theoretical viewpoint, the labelled rules from [13] are not simpler than the rules introduced in Section 4. Moreover, even if a general termination argument for all the systems is presented, complexity issues are not analysed in detail.

An unlabelled sequent calculus for $\mathbb{PCL}$ yielding an optimal PSPACE decision procedure is presented in [26]. The calculus is obtained by closing one step rules by all possible cuts and by adding the following rule, specific for $\mathbb{PCL}$:

$$\frac{\Delta_M(\nu(M)) \text{ for each } M \in \mathfrak{S}_{\Gamma_0}}{\underbrace{\{\neg(A_i > B_i) \mid i \in I\}, (A_0 > B_0)}_{\equiv:\, \Gamma_0}}\ (\mathcal{S})$$

where $\mathfrak{S}_{\Gamma_0}$ is a set of special linear pre-orders associated to the sequent Γ_0, and $\Delta_M(\nu(M))$ contains the formulas obtained by decomposing in a specific way the conditionals $(A_i > B_i)$ in the antecedent of Γ_0 according to each linear order $M \in \mathfrak{S}_{\Gamma_0}$. We refer to [26] for more a detailed explanation. The resulting proof system is undoubtedly significant, but the rules have a highly combinatorial nature and are overly complicated. In particular, a non-trivial calculation (although the algorithm is polynomial) is needed to obtain one backward instance of the $(\mathcal{S})$-rule for a given sequent.

Recently, a resolution calculus for $\mathbb{PCL}$ has been proposed in [18]. The calculus does not make use of labels, nor of any additional structure; it relies however on a non-trivial pre-processing of formulas (including renaming of subformulas and addition of propositional constants) in order to transform a formula into a suitable set of clauses to which the resolution rules can be applied.

As a difference with Lewis' logics[19], it is remarkable that today, 40 years since preferential logics has been introduced, no *standard* unlabelled sequent calculi for $\mathbb{PCL}$ or its extensions have been found, where by a standard calculus we mean a proof system with a fixed finite number of rules, each with a fixed finite number of premisses.

[19]Refer to [11, 12] for recently proposed non-labelled calculi for Lewis' logics.

Regarding labelled sequent calculi for preferential logics, from a computational viewpoint the main issue, to be explored in future work, is whether the calculi can be refined in order to achieve optimal complexity. This may lead to a redefinition of the semantics itself, in order to obtain sharper labelled rules, or to a modification of the structure of sequents.

8.2 Conclusions

In this paper we have studied the preferential conditional logic $\mathbb{PCL}$ and its extensions. We have first provided a natural semantics for this class of logics in terms of neighbourhood models. Neighbourhood models generalise Lewis' sphere models for counterfactual logics. We have given a *direct* proof of soundness and completeness of $\mathbb{PCL}$ and its extensions with respect to this class of models, with the exception of $\mathbb{PW}$ and $\mathbb{PWU}$. We have then presented labelled sequent calculi for all logics of the family. The calculi are modular and have standard proof-theoretical properties, the most important being cut admissibility, by which completeness of the calculi easily follows. We have tackled the issue of termination of the calculi, with the aim of obtaining a decision procedure for each logic. For all systems, except for those containing absoluteness, we have shown that by adopting a suitable strategy, it holds that every derivation either succeeds or ends by producing a finite tree. With respect to the known complexity of the logics, the decisions procedures are not optimal, and further work is needed to obtain optimal procedures out of the labelled calculi. In future work we will study how to obtain an optimal decision procedure for the logics.

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Appendix

Proof of Theorem 5.9. We have to show that the inference rules of $\mathcal{H}_{\mathsf{PCL}}$ are admissible, and that the axioms of $\mathcal{H}_{\mathsf{PCL}}$ and its extensions are derivable. By means of example, we show admissibility of (RCEA) and (RCK) in **G3P.CL**, derivability of axioms (CM) in **G3P.CL**, of (N) in **G3P.N**, of (T) in **G3P.T** and of (U_1) in **G3P.U**. More derivation examples can be found in [10].

In the following derivations, wk^* denotes several occurrences of $\mathsf{wk_L}$ and $\mathsf{wk_R}$. Weakening rules are applied several times in the same derivation branch to diminish the number of formulas and improve readability. Moreover, st denotes an application of Proposition 5.5 (substitution), inv denotes an application of 2 of Lemma 5.7 (invertibility of the rules) and init here denotes generalized initial sequents, proved to be derivable in Lemma 5.6 (generalized initial sequents).

For (RCEA), assume $\vdash A \to B$ and $\vdash A \to B$. Then, we have derivations of sequents $\Rightarrow x : A \to B$ and $\Rightarrow x : B \to A$, for some label x. We show how to obtain a derivation of sequent $\Rightarrow x : (A > C) \to (B > C)$ (the converse direction can be proved in a similar way). In the following derivation, $\Gamma = \{c \subseteq a, c \in N(x)\}$ and $D = x \Vdash_a B|C$.

$$
\begin{prooftree}
\AxiomC{$\Rightarrow x : B \to A$}
\dottedLine
\RightLabel{st}
\UnaryInfC{$\Rightarrow k : B \to A$}
\dottedLine
\RightLabel{inv}
\UnaryInfC{$k : B \Rightarrow k : A$}
\RightLabel{wk^*}
\UnaryInfC{$k \in a, a \Vdash^\exists B, k : B \Rightarrow k : A$}
\RightLabel{$\mathsf{R} \Vdash^\exists$}
\UnaryInfC{$k \in a, a \Vdash^\exists B, \Rightarrow k : A$}
\RightLabel{$\mathsf{L} \Vdash^\exists$}
\UnaryInfC{$a \Vdash^\exists B \Rightarrow a \Vdash^\exists A$}
\RightLabel{wk^*}
\UnaryInfC{$a \in N(x), a \Vdash^\exists B, x : A > C \Rightarrow x \Vdash_a B|C, a \Vdash^\exists A$}
\AxiomC{$\Rightarrow x : A \to B$}
\dottedLine
\RightLabel{st}
\UnaryInfC{$\Rightarrow y : A \to B$}
\dottedLine
\RightLabel{inv}
\UnaryInfC{$y : A \Rightarrow y : B$}
\RightLabel{wk^*}
\UnaryInfC{$y \in c, y : A \Rightarrow c \Vdash^\exists B, y : B$}
\RightLabel{$\mathsf{R} \Vdash^\exists$}
\UnaryInfC{$y \in c, y : A \Rightarrow c \Vdash^\exists B$}
\RightLabel{$\mathsf{L} \Vdash^\exists$}
\UnaryInfC{$c \Vdash^\exists A \Rightarrow c \Vdash^\exists B$}
\RightLabel{wk^*}
\UnaryInfC{$\Gamma, c \Vdash^\exists A, c \Vdash^\forall A \to C \Rightarrow D, c \Vdash^\exists B$}
\AxiomC{}
\RightLabel{init}
\UnaryInfC{$z : C \Rightarrow z : C$}
\AxiomC{$\Rightarrow x : B \to A$}
\dottedLine
\RightLabel{st}
\UnaryInfC{$\Rightarrow z : B \to A$}
\dottedLine
\RightLabel{inv}
\UnaryInfC{$z : B \Rightarrow z : A$}
\RightLabel{$\mathsf{L} \to$}
\BinaryInfC{$z : A \to C, z : A \Rightarrow z : C$}
\RightLabel{$\mathsf{R} \to$}
\UnaryInfC{$z : A \to C \Rightarrow z : A \to C$}
\RightLabel{wk^*}
\UnaryInfC{$z \in c, z : A \to C, c \Vdash^\forall A \to C \Rightarrow z : B \to C$}
\RightLabel{$\mathsf{L} \Vdash^\forall$}
\UnaryInfC{$z \in c, c \Vdash^\forall A \to C \Rightarrow z : B \to C$}
\RightLabel{$\mathsf{R} \Vdash^\forall$}
\UnaryInfC{$c \Vdash^\forall A \to C \Rightarrow c \Vdash^\forall B \to C$}
\RightLabel{wk^*}
\UnaryInfC{$\Gamma, c \Vdash^\exists A, c \Vdash^\forall A \to C \Rightarrow D, c \Vdash^\forall B \to C$}
\RightLabel{$\mathsf{R}|$}
\BinaryInfC{$c \subseteq a, c \in N(x), c \Vdash^\exists A, c \Vdash^\forall A \to C \Rightarrow x \Vdash_a B|C$}
\RightLabel{$\mathsf{L}|$}
\UnaryInfC{$x \Vdash_a A|C \Rightarrow x \Vdash_a B|C$}
\RightLabel{wk^*}
\UnaryInfC{$x \Vdash_a A|C, a \in N(x), a \Vdash^\exists B, x : A > C \Rightarrow x \Vdash_a B|C$}
\RightLabel{$\mathsf{L} >$}
\BinaryInfC{$a \in N(x), a \Vdash^\exists B, x : A > C \Rightarrow x \Vdash_a B|C$}
\RightLabel{$\mathsf{R} >$}
\UnaryInfC{$x : A > C \Rightarrow x : B > C$}
\RightLabel{$\mathsf{R} \to$}
\UnaryInfC{$\Rightarrow x : (A > C) \to (B > C)$}
\end{prooftree}
$$

For (RCK), assume $\vdash A \to B$. Thus, we have a derivation of sequent $\Rightarrow x : A \to B$. We show how to derive sequent $\Rightarrow x : (C > A) \to (C > B)$. In the following derivation, $\Gamma = \{b \subseteq a, b \in N(x)\}$ and $D = x \Vdash_a C|B$.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{init}
\UnaryInfC{$a \in N(x), a \Vdash^{\exists} C, x : C > A \Rightarrow x \Vdash_a C|B, a \Vdash^{\exists} C$}
\AxiomC{}
\RightLabel{init}
\UnaryInfC{$\Gamma, b \Vdash^{\exists} C, b \Vdash^{\forall} C \to A, \Rightarrow D, b \Vdash^{\exists} C$}
\AxiomC{}
\RightLabel{init}
\UnaryInfC{$y : C \Rightarrow y : C$}
\AxiomC{$\Rightarrow x : A \to B$}
\RightLabel{st}
\dottedLine
\UnaryInfC{$\Rightarrow y : A \to B$}
\RightLabel{inv}
\dottedLine
\UnaryInfC{$y : A \Rightarrow y : B$}
\RightLabel{L $\to$}
\BinaryInfC{$y : C \to A, y : C \Rightarrow y : B$}
\RightLabel{R $\to$}
\UnaryInfC{$y : C \to A \Rightarrow y : C \to B$}
\RightLabel{wk*}
\UnaryInfC{$y \in b, b \Vdash^{\forall} C \to A, y : C \to A \Rightarrow y : C \to B$}
\RightLabel{L $\Vdash^{\forall}$}
\UnaryInfC{$y \in b, b \Vdash^{\forall} C \to A \Rightarrow y : C \to B$}
\RightLabel{R $\Vdash^{\forall}$}
\UnaryInfC{$b \Vdash^{\forall} C \to A \Rightarrow b \Vdash^{\forall} C \to B$}
\RightLabel{wk*}
\UnaryInfC{$\Gamma, b \Vdash^{\exists} C, b \Vdash^{\forall} C \to A, \Rightarrow D, b \Vdash^{\forall} C \to B$}
\RightLabel{R$|$}
\BinaryInfC{$b \subseteq a, b \in N(x), b \Vdash^{\exists} C, b \Vdash^{\forall} C \to A, \Rightarrow x \Vdash_a C|B$}
\RightLabel{L$|$}
\UnaryInfC{$x \Vdash_a C|A \Rightarrow x \Vdash_a C|B$}
\RightLabel{wk*}
\UnaryInfC{$x \Vdash_a C|A, a \in N(x), a \Vdash^{\exists} C, x : C > A \Rightarrow x \Vdash_a C|B$}
\RightLabel{L $>$}
\BinaryInfC{$a \in N(x), a \Vdash^{\exists} C, x : C > A \Rightarrow x \Vdash_a C|B$}
\RightLabel{R $>$}
\UnaryInfC{$x : C > A \Rightarrow x : C > B$}
\RightLabel{R $\to$}
\UnaryInfC{$\Rightarrow x : (C > A) \to (C > B)$}
\end{prooftree}
$$

We now show how to derive axiom (CM). For reasons of space, we have omitted to report in the premiss of rules the formulas to which no other inference rules are applied. Moreover, we have omitted the following (derivable) three sequents.

Left premiss of R$|$:
$c \subseteq a, c \subseteq b, c \Vdash^{\exists} A, c \Vdash^{\exists} A, c \Vdash^{\forall} A \to C, b \Vdash^{\exists} A, b \Vdash^{\exists} A \to B, a \Vdash^{\exists} A \wedge B, x : A > B, x : A > C \Rightarrow x \Vdash_a A \wedge B|C, c \Vdash^{\exists} A \wedge B$

Left premiss of L $>$ (upper occurrence):
$b \in N(x), b \subseteq a, a \in N(x), b \Vdash^{\exists} A, b \Vdash^{\exists} A \to B, a \Vdash^{\exists} A \wedge B, x : A > B, x : A > C \Rightarrow x \Vdash_a A \wedge B|C, b \Vdash^{\exists} A$

Left premiss of L $>$ (lower occurrence):
$a \in N(x), a \Vdash^{\exists} A \wedge B, x : A > B, x : A > C \Rightarrow x \Vdash_a A \wedge B|C, a \Vdash^{\exists} A$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{init}
\UnaryInfC{$\dots, y : C \Rightarrow y : C, \dots$}
\AxiomC{}
\RightLabel{init}
\UnaryInfC{$\dots, y : A, y : B \Rightarrow y : A, \dots$}
\RightLabel{L$\wedge$}
\UnaryInfC{$\dots, y : A \wedge B \Rightarrow y : A, \dots$}
\RightLabel{L $\to$}
\BinaryInfC{$y \in c, c \subseteq a, c \subseteq b, c \Vdash^{\exists} A, y : A \to C, b \Vdash^{\exists} A, b \Vdash^{\forall} A \to B, a \Vdash^{\exists} A \wedge B, y : A \wedge B, \dots \Rightarrow \dots, y : C$}
\RightLabel{L $\Vdash^{\forall}$}
\UnaryInfC{$y \in c, c \subseteq a, c \subseteq b, c \Vdash^{\exists} A, c \Vdash^{\forall} A \to C, b \Vdash^{\exists} A, b \Vdash^{\exists} A \to B, a \Vdash^{\exists} A \wedge B, y : A \wedge B, \dots \Rightarrow \dots, y : C$}
\RightLabel{R $\to$}
\UnaryInfC{$y \in c, c \subseteq a, c \subseteq b, c \Vdash^{\exists} A, c \Vdash^{\forall} A \to C, b \Vdash^{\exists} A, b \Vdash^{\exists} A \to B, a \Vdash^{\exists} A \wedge B, \dots \Rightarrow \dots, y : A \wedge B \to C$}
\RightLabel{R $\Vdash^{\forall}$}
\UnaryInfC{$c \subseteq a, c \subseteq b, c \Vdash^{\exists} A, c \Vdash^{\forall} A \to C, b \Vdash^{\exists} A, b \Vdash^{\exists} A \to B, a \Vdash^{\exists} A \wedge B, \dots \Rightarrow \dots, c \Vdash^{\forall} A \wedge B \to C$}
\RightLabel{R$|$}
\UnaryInfC{$c \subseteq a, c \subseteq b, c \Vdash^{\exists} A, c \Vdash^{\exists} A, c \Vdash^{\forall} A \to C, b \Vdash^{\exists} A, b \Vdash^{\exists} A \to B, a \Vdash^{\exists} A \wedge B, x : A > B, x : A > C \Rightarrow x \Vdash_a A \wedge B|C$}
\RightLabel{tr}
\UnaryInfC{$c \in N(x), c \subseteq b, b \in N(x), b \subseteq a, a \in N(x), c \Vdash^{\exists} A, c \Vdash^{\exists} A \to C, b \Vdash^{\exists} A, b \Vdash^{\exists} A \to B, a \Vdash^{\exists} A \wedge B, x : A > B, x : A > C \Rightarrow x \Vdash_a A \wedge B|C$}
\RightLabel{L$|$}
\UnaryInfC{$x \Vdash_b A|C, b \in N(x), b \subseteq a, a \in N(x), b \Vdash^{\exists} A, b \Vdash^{\exists} A \to B, a \Vdash^{\exists} A \wedge B, x : A > B, x : A > C \Rightarrow x \Vdash_a A \wedge B|C$}
\RightLabel{L $>$}
\UnaryInfC{$b \in N(x), b \subseteq a, a \in N(x), b \Vdash^{\exists} A, b \Vdash^{\exists} A \to B, a \Vdash^{\exists} A \wedge B, x : A > B, x : A > C \Rightarrow x \Vdash_a A \wedge B|C$}
\RightLabel{L$|$}
\UnaryInfC{$x \Vdash_a A|B, a \in N(x), a \Vdash^{\exists} A \wedge B, x : A > B, x : A > C \Rightarrow x \Vdash_a A \wedge B|C$}
\RightLabel{L $>$}
\UnaryInfC{$a \in N(x), a \Vdash^{\exists} A \wedge B, x : A > B, x : A > C \Rightarrow x \Vdash_a A \wedge B|C$}
\RightLabel{R $>$}
\UnaryInfC{$x : A > B, x : A > C \Rightarrow x : (A \wedge B) > C$}
\RightLabel{L$\wedge$}
\UnaryInfC{$x : (A > B) \wedge (A > C) \Rightarrow x : (A \wedge B) > C$}
\RightLabel{R $\to$}
\UnaryInfC{$\Rightarrow x : (A > B) \wedge (A > C) \to ((A \wedge B) > C)$}
\end{prooftree}
$$

We now show how the axiom (N) $\neg(\top > \bot)$ can be derived. Using only the primitive connectives of the language, the axiom becomes $((p \vee (p \to \bot)) > \bot) \to \bot$. Again, not all formulas are shown in the derivation.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{init}
\UnaryInfC{$y \in a, \dots y : p \Rightarrow \dots, y : p, y : \bot$}
\RightLabel{$\mathsf{R}\rightarrow$}
\UnaryInfC{$y \in a, \dots \Rightarrow \dots, y : p, y : p \to \bot$}
\RightLabel{$\mathsf{R}\vee$}
\UnaryInfC{$y \in a, \dots \Rightarrow \dots, y : p \vee (p \to \bot)$}
\RightLabel{$\mathsf{R} \Vdash^{\exists}$}
\UnaryInfC{$y \in a, \dots \Rightarrow x : \bot, y : p \vee (p \to \bot)$}
\RightLabel{0}
\UnaryInfC{$\dots \Rightarrow x : \bot, a \Vdash^{\exists} p \vee (p \to \bot)$}
\AxiomC{}
\RightLabel{$\bot_{\mathsf{L}}$}
\UnaryInfC{$\dots, z : \bot \Rightarrow \dots$}
\AxiomC{}
\RightLabel{init}
\UnaryInfC{$\dots z : p \Rightarrow z : p, x : \bot$}
\RightLabel{$\mathsf{R}\rightarrow$}
\UnaryInfC{$\dots \Rightarrow z : p, z : p \to \bot, x : \bot$}
\RightLabel{$\mathsf{R}\vee$}
\UnaryInfC{$\dots \Rightarrow z : p \vee (p \to \bot), x : \bot$}
\RightLabel{$\mathsf{L}\rightarrow$}
\BinaryInfC{$z \in c, \dots, z : (p \vee (p \to \bot)) \to \bot \Rightarrow x : \bot$}
\RightLabel{$\mathsf{L} \Vdash^{\forall}$}
\UnaryInfC{$c \in N(x), c \subseteq a, c \Vdash^{\exists} p \vee (p \to \bot), c \Vdash^{\forall} (p \vee (p \to \bot)) \to \bot, \dots \Rightarrow x : \bot$}
\RightLabel{$\mathsf{L}|$}
\UnaryInfC{$x \Vdash_a p \vee (p \to \bot) | \bot, \dots \Rightarrow x : \bot$}
\RightLabel{$\mathsf{L}>$}
\BinaryInfC{$a \in N(x), (p \vee (p \to \bot)) > \bot \Rightarrow x : \bot$}
\RightLabel{N}
\UnaryInfC{$(p \vee (p \to \bot)) > \bot \Rightarrow x : \bot$}
\RightLabel{$\mathsf{R}\rightarrow$}
\UnaryInfC{$\Rightarrow x : ((p \vee (p \to \bot)) > \bot) \to \bot$}
\end{prooftree}
$$

We show how a derivation of axiom (T) can be obtained. The left premiss of $\mathsf{L}>$ is the derivable sequent $x \in a, a \in N(x), x : A, x : A > \bot \Rightarrow x : \bot, a \Vdash^{\exists} A$, not shown for reasons of space.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{$\bot_{\mathsf{L}}$}
\UnaryInfC{$\dots, y : \bot \Rightarrow \dots$}
\AxiomC{}
\RightLabel{init}
\UnaryInfC{$\dots, y : A \Rightarrow y : A, \dots$}
\RightLabel{$\mathsf{L}\rightarrow$}
\BinaryInfC{$y \in b, y : A, b \in N(x), b \subseteq a, b : A, y : A \to \bot, b \Vdash^{\forall} A \to \bot, x \in a, a \in N(x), x : A > \bot \Rightarrow x : \bot$}
\RightLabel{$\mathsf{L} \Vdash^{\forall}$}
\UnaryInfC{$y \in b, b \in N(x), b \subseteq a, x \in a, a \in N(x), y : A, b \Vdash^{\forall} A \to \bot, x : A > \bot \Rightarrow x : \bot$}
\RightLabel{$\mathsf{L} \Vdash^{\exists}$}
\UnaryInfC{$b \in N(x), b \subseteq a, x \in a, a \in N(x), b \Vdash^{\exists} A, b \Vdash^{\forall} A \to \bot, x : A > \bot \Rightarrow x : \bot$}
\RightLabel{$\mathsf{L}|$}
\UnaryInfC{$x \Vdash_a A | \bot, x \in a, a \in N(x), x : A, x : A > \bot, \Rightarrow x : \bot$}
\RightLabel{$\mathsf{L}>$}
\UnaryInfC{$x \in a, a \in N(x), x : A, x : A > \bot \Rightarrow x : \bot$}
\RightLabel{T}
\UnaryInfC{$x : A, x : A > \bot \Rightarrow x : \bot$}
\RightLabel{$\mathsf{R}\rightarrow$}
\UnaryInfC{$x : A \Rightarrow x : (A > \bot) \to \bot$}
\RightLabel{$\mathsf{R}\rightarrow$}
\UnaryInfC{$\Rightarrow x : A \to ((A > \bot) \to \bot$}
\end{prooftree}
$$

Finally, here follows the derivation of axiom (U_1) $(\neg A > \bot) \to (\neg(\neg A > \bot) > \bot)$ which, reformulated using only the primitive connectives of the language, becomes $((A \to \bot) > \bot) \to (((A \to \bot) > \bot) \to \bot) > \bot$. We have omitted writing the right premisses of the three occurrences of $\mathsf{L}\rightarrow$ in the derivation, derivable by $\bot_{\mathsf{L}}$.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{init}
\UnaryInfC{$z \in c, \dots, z : A \to \bot \Rightarrow \dots, z : A \to \bot$}
\RightLabel{$\mathsf{R} \Vdash^{\exists}$}
\UnaryInfC{$z \in c, \dots, z : A \to \bot \Rightarrow \dots, c \Vdash^{\exists} A \to \bot$}
\AxiomC{}
\RightLabel{init}
\UnaryInfC{$k \in d, d \subseteq c, d \in N(x), \dots, z : A \Rightarrow \dots, k : A, z : \bot$}
\RightLabel{$\mathsf{R}\rightarrow$}
\UnaryInfC{$k \in d, d \subseteq c, d \in N(x), \dots \Rightarrow \dots, k : A, z : A \to \bot$}
\RightLabel{$\mathsf{L}\rightarrow$}
\UnaryInfC{$k \in d, d \subseteq c, d \in N(x), z : (A \to \bot) \to \bot, \dots \Rightarrow \dots, k : A$}
\RightLabel{$\mathsf{L} \Vdash^{\forall}$}
\UnaryInfC{$k \in d, d \subseteq c, d \in N(x), d \Vdash^{\forall} (A \to \bot) \to \bot, \dots \Rightarrow \dots, k : A$}
\RightLabel{$\mathsf{L}\rightarrow$}
\UnaryInfC{$k \in d, d \subseteq c, d \in N(x), k : A \to \bot, d \Vdash^{\forall} (A \to \bot) \to \bot, \dots \Rightarrow \dots$}
\RightLabel{$\mathsf{L} \Vdash^{\exists}$}
\UnaryInfC{$d \subseteq c, d \in N(x), d \Vdash^{\exists} A \to \bot, d \Vdash^{\forall} (A \to \bot) \to \bot, \dots \Rightarrow \dots$}
\RightLabel{$\mathsf{L}|$}
\UnaryInfC{$x \Vdash_c A \to \bot \mid \bot, \dots \Rightarrow \dots$}
\RightLabel{$\mathsf{L}>$}
\BinaryInfC{$z \in c, c \in N(x), z \in b, y \in a, a \in N(x), b \in N(y), z : A \to \bot, x : (A \to \bot) > \bot \Rightarrow \dots$}
\RightLabel{U_1}
\UnaryInfC{$z \in b, y \in a, a \in N(x), b \in N(y), z : A \to \bot, x : (A \to \bot) > \bot \Rightarrow x \Vdash_a ((A \to \bot) > \bot) \to \bot | \bot, y \Vdash_b A \to \bot | \bot$}
\RightLabel{$\mathsf{L} \Vdash^{\exists}$}
\UnaryInfC{$y \in a, a \in N(x), b \in N(y), b \Vdash^{\exists} A \to \bot, x : (A \to \bot) > \bot \Rightarrow x \Vdash_a ((A \to \bot) > \bot) \to \bot | \bot, y \Vdash_b A \to \bot | \bot$}
\RightLabel{$\mathsf{R}>$}
\UnaryInfC{$y \in a, a \in N(x), x : (A \to \bot) > \bot \Rightarrow x \Vdash_a ((A \to \bot) > \bot) \to \bot | \bot, y : (A \to \bot) > \bot$}
\RightLabel{$\mathsf{L}\rightarrow$}
\UnaryInfC{$y \in a, a \in N(x), y : ((A \to \bot) > \bot) \to \bot, x : (A \to \bot) > \bot \Rightarrow x \Vdash_a ((A \to \bot) > \bot) \to \bot | \bot$}
\RightLabel{$\mathsf{L} \Vdash^{\exists}$}
\UnaryInfC{$a \in N(x), a \Vdash^{\exists} ((A \to \bot) > \bot) \to \bot, x : (A \to \bot) > \bot \Rightarrow x \Vdash_a ((A \to \bot) > \bot) \to \bot | \bot$}
\RightLabel{$\mathsf{R}>$}
\UnaryInfC{$x : (A \to \bot) > \bot \Rightarrow x : (((A \to \bot) > \bot) \to \bot) > \bot$}
\RightLabel{$\mathsf{R}\rightarrow$}
\UnaryInfC{$\Rightarrow x : ((A \to \bot) > \bot) \to (((A \to \bot) > \bot) \to \bot) > \bot$}
\end{prooftree}
$$

$\square$